PHYS 1901: OSCILLATIONS, WAVES & CHAOS
Lecture Notes (Part 1): Using complex algebra to solve DEs

1 Simple Harmonic Oscillator

We want to solve the differential equation

$$m\frac{d^2x}{dt^2} + kx = 0. \tag{1}$$

Instead, let's solve the equation

$$m\frac{d^2z}{dt^2} + kz = 0, \tag{2}$$

where z is complex. It is not hard to see that if z is a solution of Equation 2 then $\Re(z)$ is a solution of Equation 1 (see the note below). We guess the solution of Equation 2 to be

$$z = Ae^{i(\omega t+\phi)}.$$

The derivatives of z are

$$\frac{dz}{dt} = i\omega z$$

and

$$\begin{aligned} \frac{d^2z}{dt^2} &= (i\omega)^2 z \\ &= -\omega^2 z. \end{aligned}$$

Putting these into Equation 2 gives

$$-m\omega^2 z + kz = 0.$$

Hence, z is a solution of Equation 2 provided

$$-m\omega^2 + k = 0,$$

in other words, provided

$$\omega^2 = k/m.$$

We therefore conclude that, provided $\omega^2 = k/m$, then $\Re(z) = A\cos(\omega t + \phi)$ is a solution of Equation 1. This is exactly the same solution that we found previously.

Note:

We have used the fact that if z is a solution of Equation 2 then $\Re(z)$ is a solution of Equation 1. We should justify this. Write $z(t) = x(t) + iy(t)$, where we show the time dependence explicitly. The definition of the derivative of z is

$$\begin{aligned} \frac{dz}{dt} &= \lim_{\delta t\to 0}\frac{z(t+\delta t) - z(t)}{\delta t} \\ &= \lim_{\delta t\to 0}\left[\frac{x(t+\delta t) - x(t)}{\delta t} + i\frac{y(t+\delta t) - y(t)}{\delta t}\right] \\ &= \lim_{\delta t\to 0}\frac{x(t+\delta t) - x(t)}{\delta t} + i\lim_{\delta t\to 0}\frac{y(t+\delta t) - y(t)}{\delta t} \\ &= \frac{dx}{dt} + i\frac{dy}{dt}. \end{aligned}$$

Similarly,

$$\frac{d^2z}{dt^2} = \frac{d^2x}{dt^2} + i\frac{d^2y}{dt^2}$$

and so the result follows because Equation 1 is *linear* (it only contains the first power of z and its derivatives).

2 Damped Harmonic Oscillator

We now want to add a damping term proportional to speed, which means solving the differential equation

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0. \tag{3}$$

As before, we first solve the equation

$$m\frac{d^2z}{dt^2} + b\frac{dz}{dt} + kz = 0, \tag{4}$$

where z is complex. Once again, if z is a solution of Equation 4 then $\Re(z)$ is a solution of Equation 3. This follows because the differential equations is *linear* (z and its derivatives only appear in the zeroth or first power).

We guess the solution of Equation 4 to be

$$\begin{aligned} z &= Ae^{-Bt}e^{i(\omega' t+\phi)} \\ &= Ae^{[(-B+i\omega')t+i\phi]}. \end{aligned}$$

The derivatives of z are

$$\frac{dz}{dt} = (-B + i\omega')z$$

and

$$\begin{aligned} \frac{d^2z}{dt^2} &= (-B + i\omega')^2 z \\ &= (B^2 - {\omega'}^2 - 2B\omega' i)z. \end{aligned}$$

Putting these into Equation 4 gives

$$m(B^2 - {\omega'}^2 - 2B\omega' i)z + b(-B + i\omega')z + kz = 0.$$

Hence, z is a solution of Equation 4 provided

$$m(B^2 - {\omega'}^2 - 2B\omega' i) + b(-B + i\omega') + k = 0.$$

This will be true if both the real and imaginary parts of the left hand side are zero, which gives us:

$$m(B^2 - {\omega'}^2) + -bB + k = 0 \tag{5}$$

and

$$-2mB\omega' + b\omega' = 0.. \tag{6}$$

Equation 6 gives

$$B = \frac{b}{2m}, \tag{7}$$

which we then substitute into Equation 5 to obtain (after rearranging)

$${\omega'}^2 = \frac{k}{m} - \frac{b^2}{4m^2}. \tag{8}$$

We therefore conclude that z is indeed a solution of Equation 4, provided B and ω' satisfy Equations 7 and 8, respectively.

The real part of z is $\Re(z) = Ae^{-Bt}\cos(\omega' t + \phi)$, which is a solution of Equation 3 under the same conditions.

3 Driven Damped Harmonic Oscillator

Finally, we add a sinusoidal driving term. The angular frequency, ω_d, of this driving can be different from the natural frequency of the unforced oscillator, which we have been referring to as ω. We have to solve this equation:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_{\max}\cos(\omega_d t). \tag{9}$$

where $F_{\max}$ is the amplitude of the driving force. As before, we first solve the equation

$$m\frac{d^2z}{dt^2} + b\frac{dz}{dt} + kz = F_{\max}e^{i\omega_d t}, \tag{10}$$

where z is complex. Once again, if z is a solution of Equation 10 then $\Re(z)$ is a solution of Equation 9. This follows because the differential equations is *linear* (z and its derivatives only appear in the zeroth or first power).

The transient solutions are complicated and depend on the initial conditions, but in steady state (once things have settled down) it turns out that the solution is an oscillation with the frequency of the driving force. We therefore guess the solution of Equation 10 to be

$$z = Ae^{i(\omega_d t + \phi)}.$$

The derivatives of z are

$$\frac{dz}{dt} = (i\omega_d)z$$

and

$$\frac{d^2z}{dt^2} = -\omega_d^2 z.$$

Putting these into Equation 10 gives

$$-m\omega_d^2 z + ib\omega_d z + kz = \frac{F_{\max}}{A}e^{-i\phi}z.$$

Hence, z is a solution of Equation 10 provided

$$k - m\omega_d^2 + ib\omega_d = \frac{F_{\max}}{A}e^{-i\phi}. \tag{11}$$

We can equate real and imaginary parts, which gives two equations for the two unknowns (A and ϕ). However, it is easier to equate the modulus and phase of boths sides of Equation 11. In particular, we are interested in the amplitude of the oscillation. Equating the squared modulus of boths sides of Equation 11 gives us:

$$(k - m\omega_d^2)^2 + b^2\omega_d^2 = \left(\frac{F_{\max}}{A}\right)^2 \tag{12}$$

and so we arrive at

$$A = \frac{F_{\max}}{\sqrt{(k - m\omega_d^2)^2 + b^2\omega_d^2}}. \tag{13}$$

We have therefore derived Equation (13.46) in Section 13.8 of the textbook by Young & Freedman (11th & 12th Editions).