Solar and Space Physics - Assignment 2

Question 1:

a) Substitute $\mathbf{J} = \mathbf{\nabla}\times\mathbf{B}/\mu_0$ into the MHD equation of motion (3.32) with $\rho = 0$ and neglect the gravity term, to produce

$$\eta\left[\frac{\partial\mathbf{U}}{\partial t} + (\mathbf{U}\cdot\mathbf{\nabla})\mathbf{U}\right] = -\nabla p + \frac{(\mathbf{\nabla}\times\mathbf{B})\times\mathbf{B}}{\mu_0}\,.$$

Assume that the background magnetic field $\mathbf{B}_0$ and density η_0 are uniform and steady, and that there are no currents flowing, no fluid motion and no electric fields. The next step is to assume small-amplitude (linear) plane wave solutions, with the wavevector $\mathbf{k}$ parallel to the x-axis ($\mathbf{k} = k\hat{\mathbf{x}}$); for instance, $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1\exp[i(\omega t - kx)]$, where $\mathbf{B}_1 \ll \mathbf{B}_0$. Explain why the convective derivative term $(\mathbf{U}\cdot\mathbf{\nabla})\mathbf{U}$ in the equation of motion may now be dropped.

b) Replace the derivatives by

$$\frac{\partial\mathbf{U}}{\partial t} \to -i\omega\mathbf{U}_1\,,\;\; \nabla p \to ikp_1\hat{\mathbf{x}}\,,\;\; \mathbf{\nabla}\cdot\mathbf{U} \to i\mathbf{k}\cdot\mathbf{U}_1\,,\;\; \mathbf{\nabla}\times\mathbf{U} \to i\mathbf{k}\times\mathbf{U}_1\,.$$

Retaining only linear terms, and using a vector identity, show that the equation of motion takes the form

$$\omega\eta_0\mathbf{U}_1 - k(\hat{\mathbf{x}}(\mathbf{B_1}\cdot\mathbf{B_0}) - B_{0x}\mathbf{B_1})/\mu_0 - kp_1\hat{\mathbf{x}} = 0\,.$$

c) Similarly, show that the mass conservation equation (3.49) and the induction equation (3.51) reduce to equations (3.52) and (3.54).

d) Relate the small amplitude variations in pressure and density (p_1 and η_1) to their unperturbed values (p_0 and η_0) using the adiabatic equation of state $p \propto \eta^\Gamma$, to show that

$$\frac{p_1}{p_0} = \frac{\Gamma\eta_1}{\eta_0}\,.$$

Substitute this expression for p_1 in the linearized equation of motion, writing $p_1 = c_s^2\eta_1$, where c_s is the sound speed, with $c_S = \sqrt{\Gamma p_0/\eta_0}$. Then eliminate η_1 from this equation using the linearized equation of mass conservation.

e) Assume the background magnetic field is in the $x-z$ plane, with $\mathbf{B}_0 = (B_0\cos\theta, 0, B_0\sin\theta)$, and eliminate $\mathbf{B}_1$ from the two remaining vector equations, to show that the three (scalar) equations relating components of $\mathbf{U}_1$ satisfy

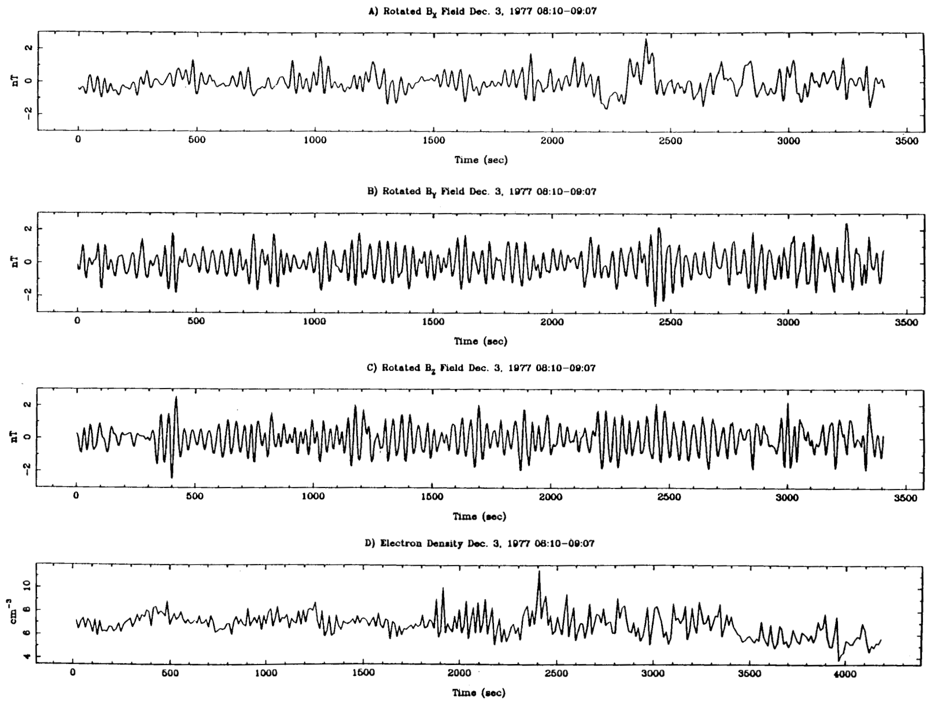
$$\begin{bmatrix} \left(\omega^2/k^2 - v_A^2\sin^2\theta - c_s^2\right) & 0 & v_A^2\sin\theta\cos\theta \\ 0 & \left(\omega^2/k^2 - v_A^2\cos^2\theta\right) & 0 \\ v_A^2\sin\theta\cos\theta & 0 & \left(\omega^2/k^2 - v_A^2\cos^2\theta\right) \end{bmatrix} \begin{bmatrix} U_{1x} \\ U_{1y} \\ U_{1z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Set the determinant of this matrix to zero and show that this yields the three solutions in equation (3.60), which correspond to the Alfvén, fast and slow mode solutions.

f) Show that Alfvén waves are incompressible (that is, the density of the fluid is unaffected by the propagating wave) via the linearized mass conservation equation.

g) Show that in the cold plasma limit ($\beta \ll 1 \Rightarrow c_s \ll v_A$) fast mode waves become magnetoacoustic waves. Demonstrate that magnetoacoustic waves are compressible.

h) The following figure shows spacecraft data of MHD waves in Earth's foreshock at a time when the ambient magnetic field and density were nearly constant. The first three panels show the x, y and z components of magnetic field, which have been rotated so that the background magnetic field is in the x-direction. The bottom panel shows the electron density. The periodic fluctuations in these data correspond to MHD waves. Comment on whether or not these waves could be Alfvén waves. Justify your answer.



Question 2:

For an unmagnetized shock ($\mathbf{B} = 0$),

$$M^2 = r/a,$$

where $M = u_1/c_1 s$ is the Mach number, r is the compression ratio of the shock, and $2a = \Gamma + 1 - r(\Gamma - 1)$, where Γ is the adiabatic index. Use this relation and the jump conditions to show that the pressure increase across an unmagnetized shock is given by

$$\frac{P_2}{P_1} = \frac{2\Gamma}{\Gamma+1}M^2 - \frac{\Gamma-1}{\Gamma+1}.$$

What is the maximum value of the pressure downstream of a strong shock?