Impact of a ball on a surface with tangential compliance

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A simple model is presented to explain how the spin of a ball can be enhanced when the ball is incident obliquely on a flexible surface. The mechanism involves tangential distortion of the surface and a return of the elastic energy stored in the surface via the action of the static friction force on the ball. As an example, we consider the enhancement of the spin that is possible using an appropriate type of tennis racquet string.
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I. INTRODUCTION

We consider the motion of a ball bouncing obliquely off a flexible surface. Examples include a billiard ball bouncing off the rubber cushion of a billiard table and the bounce of a tennis ball off the strings of a tennis racquet. The rebound speed of the ball in a direction perpendicular to the surface depends on the coefficient of restitution e_y, which is the ratio of the normal component of the rebound speed to the normal component of the incident speed. The coefficient of restitution varies between 0 and 1 and is a measure of the kinetic energy lost during the collision. If there is no loss of kinetic energy, then $e_y = 1$. Experiments show that e_y can exceed 1 for an oblique impact if the surface deforms in such a way as to redirect the ball closer to the normal.[1,2] Similarly, a tennis ball bouncing off a clay court can be deflected upward by the mound of clay pushed ahead of the ball, with the result that e_y is increased when the ball is incident at glancing angles.[3]

The rebound speed of the ball in a direction parallel to the surface is more difficult to calculate, as is the spin of the ball. There are two regimes of interest. If the ball is incident at a glancing angle, it can slide across the surface for the entire impact duration. In that case, the main parameters determining the change in the tangential speed and the change in spin are the angle of incidence and the coefficient of sliding friction. If the ball is incident at angles close to the normal, or even at angles up to about 50° away from the normal, the ball will slide for only a short time interval and then grip the surface. In the latter case the change in the tangential speed of the ball and in its spin can be characterized in terms of the tangential coefficient of restitution e_x, which is a measure of the loss of energy resulting from distortion of the ball and the impact surface in a direction parallel to the impact surface.[4,5]

The mechanism by which spin is imparted to an obliquely bouncing ball is not obvious. The change in ball spin can be attributed directly to the torque arising from the friction between the ball and the surface, but the nature and origin of the friction force are not easy to understand. Measurements show that the friction force can change direction during the bounce and can even reverse direction several times.[5] The reversal in the direction of the friction force results from the fact that the ball grips the surface during the bounce, in which case sliding friction gives way to static friction. The magnitude and direction of the static friction force depend on the magnitude and direction of the elastic distortion of the ball and the surface in a direction parallel to the impact surface.

Tennis players would like to generate as much spin as possible for certain shots and are interested in knowing what type of string is best for this objective. The physics of these shots suggests that the best type of string is one that can store and return elastic energy as a result of string movement in a direction parallel to the string plane. Strings with a rough texture exert a large sliding friction force on the ball but do not return to their original position after the impact is over due to the large grip force between overlapping strings. Low friction strings tend to spring back immediately to their original position.[3]

To investigate the effect of string motion and any other elastically deformable surface, we consider the oblique impact of a ball on a surface that is allowed to move in a tangential direction. To simplify the problem, it is assumed that the surface itself is rigid, and that elastic energy is stored in an external spring connected to the surface.

II. SIMPLIFIED MODEL OF STRING MOTION

A simplified model of the interaction between a tennis ball and the strings of a racquet is shown in Fig. 1. A ball of mass m and radius r is incident obliquely at angle θ_1 on a cart of mass M that can translate freely in the x direction, but which is attached to a rigid wall via a spring having a spring constant k. The normal reaction force N on the ball is assumed to vary with time according to the relation $N = N_0 \sin(\pi t/T)$, over the time interval $0 < t < T$, where T is the impact duration. If the ball is incident without spin, or with backspin, it will commence to slide on the upper surface of the cart and experience a horizontal friction force $F = \mu N$ in the negative x direction, where μ is the coefficient of sliding friction. The cart will experience an equal and opposite friction force in the positive x direction. During the sliding phase of the bounce, the horizontal velocity v_x of the center of mass of the ball will change according to the relation $m dv_x/dt = -F$, and hence

$$v_x = v_{x1} - \frac{\mu N_0 T}{m\pi}\left(1 - \cos\frac{\pi t}{T}\right), \quad (1)$$

where v_{x1} is the initial horizontal speed of the ball. As indicated in Fig. 1, subscript 1 is used to describe ball parameters before the bounce, subscript 2 is used to describe ball parameters after the bounce, and v_x and ω describe, respectively, the x component of the ball's velocity and its angular velocity during the bounce. The angular velocity changes according to the relation $Fr = I_{cm} d\omega/dt$, where I_{cm} is the moment of inertia of the ball about an axis through its center of mass. Hence,

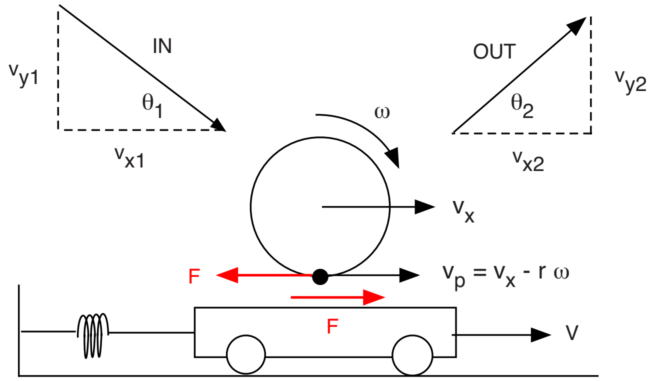

Fig. 1. A ball incident obliquely from the left on the cart causes the cart to move to the right. If elastic energy stored in the spring is returned to the ball by the action of the friction force, the ball will rebound with increased spin.

$$\omega = \omega_1 + \frac{\mu N_0 T r}{\pi I_{\rm cm}}\left(1 - \cos\frac{\pi t}{T}\right), \tag{2}$$

where ω_1 is the initial angular velocity of the ball. A point at the bottom of the ball moves in the x direction with velocity $v_p = v_x - r\omega$. The equation of motion of the cart during the sliding phase of the bounce is given by

$$M\frac{d^2X}{dt^2} = \mu N_0 \sin(\pi t/T) - kX, \tag{3}$$

where X is the horizontal displacement of the cart from its initial position. Equation (3) can be solved analytically or numerically to find the velocity $V = dX/dt$ of the cart. During the bounce, v_x decreases with time, ω increases (in the clockwise direction in Fig. 1), v_p decreases, and V increases (assuming that $V=0$ at $t=0$). If the ball is incident at a glancing angle, it will slide throughout the bounce and will bounce with $v_p > V$ at the end of the impact.

Of greater interest are cases where θ_1 is sufficiently large that the bottom of the ball comes to rest relative to the cart at some stage during the bounce, and then $v_p = V$ at that instant. If we assume that F does not change discontinuously at this time, then v_p would become less than V, and hence the direction of the friction force would reverse, which is inconsistent with the assumption that F does not change discontinuously. Alternatively, if we assume that the ball commenced rolling with $F=0$, then there would be no further change in v_p, but V would decrease due to the force exerted by the spring, in which case the ball will immediately start sliding on the cart and F will change to a nonzero value. Consequently, F must change discontinuously to a value that allows v_p to remain equal to V. The ball will continue to rotate, but the bottom of the ball will remain at rest with respect to the cart. The ball and the cart will then engage like two gears, allowing the ball to spin faster than it would if it simply rolled on a stationary surface.[6] The latter condition can be maintained provided that the required static friction force F_s remains less than μN; otherwise, the ball will start sliding again.

While $v_p = V$, $dv_x/dt - r d\omega/dt = dV/dt$, and thus

$$-\frac{F_s}{m} - \frac{F_s r^2}{I_{\rm cm}} = \frac{F_s - kX}{M}, \tag{4}$$

and hence

$$F_s = \frac{mkX}{M(1 + m/M + mr^2/I_{\rm cm})}. \tag{5}$$

The static friction force is linearly proportional to the restoring force of the spring and reverses direction when the restoring force reverses direction. The static friction acting on the ball acts to change its speed and spin according to the relations $m dv_x/dt = -F_s$ and $F_s r = I_{\rm cm} d\omega/dt$.

The equation of motion of the cart during the grip phase of the bounce is

$$\frac{d^2X}{dt^2} = -F_s\left(\frac{1}{m} + \frac{r^2}{I_{\rm cm}}\right) = -\frac{kX}{M + m\alpha/(1+\alpha)}, \tag{6}$$

where we have expressed $I_{\rm cm}$ in the form $I_{\rm cm} = \alpha m r^2$. For a tennis ball, $\alpha = 0.55$.[5] The cart therefore undergoes simple harmonic motion during the grip phase at a frequency determined by the stiffness of the spring and the mass of the cart plus a fraction of the mass of the ball. The cart and the ball do not translate as a rigid body of total mass $M+m$ during the grip phase. Rather, the bottom of the ball translates at the same speed as the cart, while the center of the ball translates at a different speed (as indicated in Fig. 1).

Near the end of the impact the normal reaction force on the ball drops to zero, and the ball may no longer be able to maintain a firm grip on the cart. If the static friction force given by Eq. (5) exceeds μN, the ball will recommence sliding on the cart. There is no instantaneous change in the magnitude or direction of the friction force if the ball starts sliding again, but the friction force is then proportional to the normal reaction force rather than the displacement of the cart. However, if the cart returns to its original position at the end of the bounce period so that F_s approaches zero as N approaches zero, then the second sliding phase can be avoided.

III. RESULTS OF MODEL CALCULATIONS

Results are presented in this section for a tennis ball with mass of $m = 57$ g and radius of $r = 33$ mm incident at speed of $v_1 = 10$ m/s at various angles on a cart with a coefficient of sliding friction of $\mu = 0.4$, which is a typical value of μ for a tennis ball sliding on smooth tennis strings. Two values of the incident ball spin are considered, $\omega_1 = 0$ and $\omega_1 = -100$ rad/s. A ball incident with backspin slides for a longer period of time than a ball incident without spin because the friction force first acts to reduce the spin to zero and then reverses the direction of spin. It is assumed, for simplicity, that the ball rebounds with $e_y = 1$, in which case the peak normal force is given by $N_0 = \pi m v_{y1}/T$, where $v_{y1} = v_1 \sin\theta_1$. T is taken to be 5 ms for the following calculations. Numerical solutions of Eqs. (1)–(6) were obtained by first solving Eq. (3) during the initial sliding stage of the bounce. The solution is continued until either the ball bounces or grips the surface. If the ball grips, then Eq. (6) is used instead of Eq. (3) to determine the subsequent motion of the cart and the ball. The solution is again continued until either the ball bounces or recommences sliding, in which case the final stage of the bounce is determined by solving Eq. (3).

The critical angle of incidence at which the ball stops sliding and grips the cart can be calculated analytically for an infinitely heavy cart where the grip condition is given by

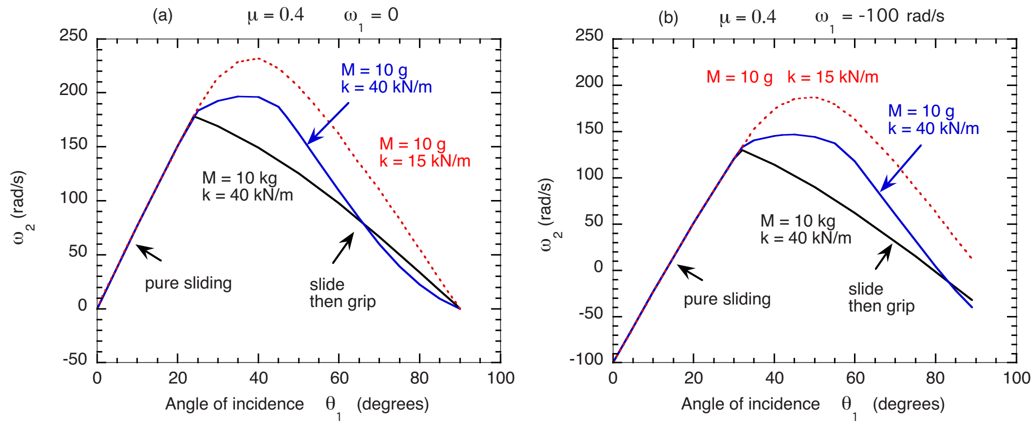

Fig. 2. Rebound spin ω_2 versus θ_1 for a ball incident at 10 m/s on a heavy cart ($M=10$ kg) or a light cart ($M=10$ g) connected to a spring with k as shown. Results are shown for cases where (a) $\omega_1=0$ and (b) $\omega_1=-100$ rad/s.

$v_{x2}=r\omega_2$, where ω_2 is the angular velocity of the ball after it bounces. In this case, we find from Eqs. (1) and (2) that

$$\tan\theta_1=\frac{(1-r\omega_1/v_{x1})}{2\mu(1+1/\alpha)}. \tag{7}$$

For a light cart, the sliding condition can be maintained over a slightly larger range of incident angles if, as in Fig. 2, the cart is moving backward by the time the ball bounces. Figure 2 shows the rebound ball spin ω_2 versus θ_1 for a spring of stiffness $k=4\times10^4$ N/m connected to either a heavy cart ($M=10$ kg) or a light cart ($M=10$ g). Also shown is the result when the 10 g cart is connected to a softer spring, with $k=1.5\times10^4$ N/m. At glancing angles of incidence, the rebound speed, spin, and angle are independent of the mass of the cart and the stiffness of the spring because the ball slides throughout the impact. If $\omega_1=0$, the ball slides throughout the bounce when $\theta_1<23.9°$ for all three conditions shown in Fig. 2(a). The sliding condition persists up to $\theta_1=26.6°$ with the softer spring. If $\omega_1=-100$ rad/s, the ball slides throughout the bounce when $\theta_1<31.0°$ [as indicated by Eq. (7)] and at slightly higher angles with the softer spring. In the pure sliding case, the bounce parameters are determined by the coefficient of sliding friction and the angle of incidence.

From Eq. (2), the rebound spin at glancing angles is given by

$$\omega_2=\omega_1+\frac{2\mu v_1\sin\theta_1}{\alpha r}. \tag{8}$$

For the parameters used to calculate the results in Fig. 2, $\omega_2=\omega_1+440.8\sin\theta_1$. When $\omega_1=0$ and $\theta_1=23.9°$, $\omega_2=178.6$ rad/s. For the heavy cart, this value is the maximum spin of the ball because ω_2 decreases as the angle of incidence increases above 23.9°. If the ball is incident with backspin, the angle of incidence can be increased in an attempt to increase the spin, but the resulting ball spin remains less than that for a ball incident without spin, as shown in Fig. 2(b).

At large angles of incidence, where the ball grips the cart, the bounce parameters depend not only on the angle of incidence but also on the mass of the cart and the spring stiffness. For a heavy cart, the outgoing ball spin decreases as the angle of incidence increases, and it decreases to zero at normal incidence. A significant increase in spin can result if the mass of the cart is less than the mass of the ball and if the period of oscillation of the cart-spring system is comparable to the impact duration of the ball. At angles of incidence where $\theta_1<60°$, the rebound spin is larger when $\omega_1=0$ than when $\omega_1=-100$ rad/s. However, when $\theta_1>60°$, the rebound spin for the 10 g cart can be larger when $\omega_1=-100$ rad/s than when $\omega_1=0$, depending on the spring stiffness. The latter result can be attributed to the fact that the ball slides for a longer time when it is incident with backspin, with the result that more elastic energy is stored in the spring. A surprising conclusion is that it may be easier, rather than more difficult, for a tennis player to return a ball with topspin if the incoming ball is spinning backward. Normally, a ball bounces off the court with topspin. However, the ball is then spinning backward relative to the receiver's racquet, meaning that the receiver needs to reverse the spin direction to return the ball with topspin.[3]

Figure 3 shows the friction force acting on the ball at the angle of incidence of 60° for the three cases shown in Fig. 2(b). When incident on the 10 kg cart, the ball slides for 2.25 ms and then grips the cart, at which time the force drops almost to zero. The motion of the ball then becomes essentially a rolling mode and bounces with $r\omega_2/v_{x2}=0.992$. The displacement of the cart is very small in this case, and the energy stored in the spring is negligible. A significantly larger displacement of the cart results when the mass of the cart is smaller than the mass of the ball. In Fig. 3(b) the cart reaches a forward speed of 3.2 m/s at 1.7 ms, at which time the ball grips the cart and the force drops from 109 to 90 N. The cart then undergoes simple harmonic motion, resulting in a reversal of the friction force at a time when the displacement of the cart is zero [as indicated by Eq. (5)]. As a result of the reversal of F, ω decreases with time. The cart has a maximum positive displacement of 4.4 mm at $t=2.3$ ms in Fig. 3(b). Near the end of the bounce, the ball recommences sliding.

In Fig. 3(c) the cart reaches a maximum forward speed of 5.0 m/s at 1.34 ms, at which time the ball grips the cart. The cart undergoes simple harmonic motion at a lower frequency than that in Fig. 3(b) due to the softer spring and has a maximum displacement of 7.7 mm at 3.1 ms. The friction force drops to zero at the end of the bounce period because the cart returns to its original position at that time. As a

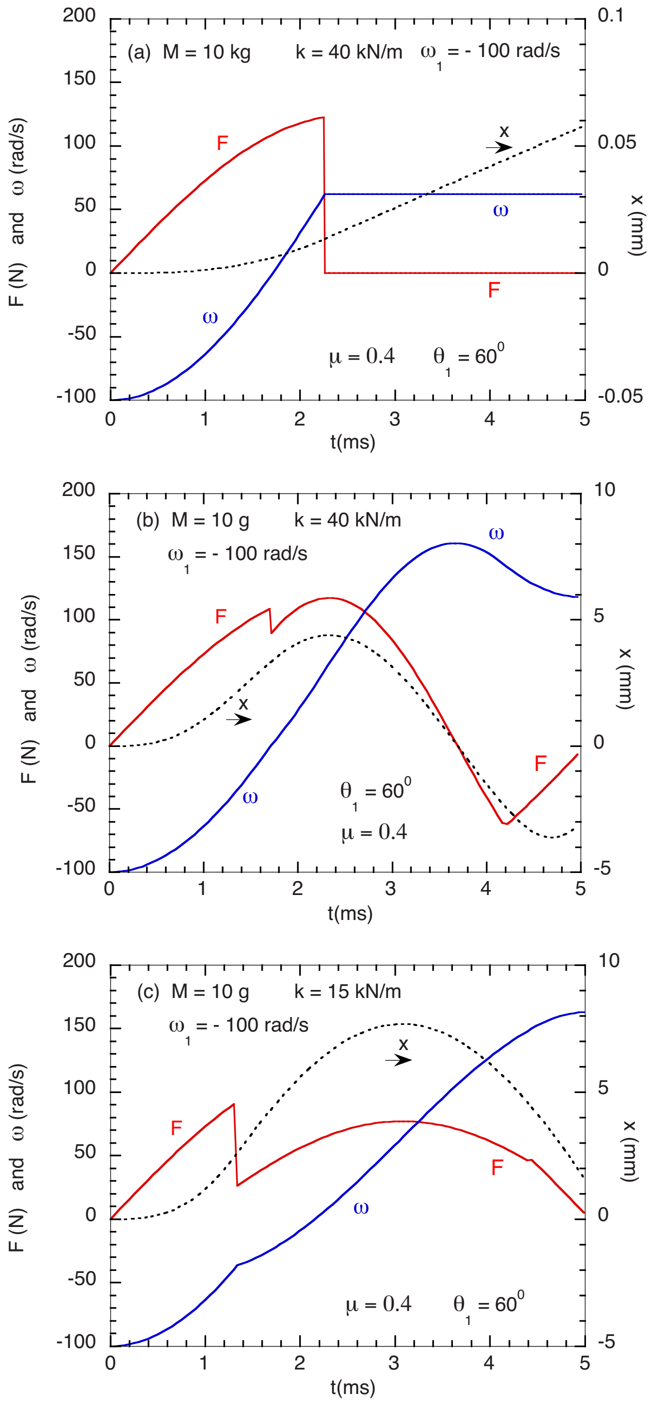

Fig. 3. The friction force F, cart displacement X, and ball spin ω versus the time when the ball is incident at $\theta_1=60°$ for the three cases shown in Fig. 2(b). A positive value of F indicates that the friction force on the ball acts in the negative x direction in Fig. 1.

result, the time-integrated force on the ball is larger than that in Fig. 3(b) and the ball rebounds with greater spin, as shown in Fig. 2.

Even larger spins can be achieved by reducing the mass of the cart below 10 g and by adjusting the stiffness of the spring so that the oscillation period of the cart matches the duration of the impact, as in Fig. 3(c). For example, the maximum ball spin using the 10 g cart is 162.8 rad/s when $\theta_1=60°$, $\omega_1=-100$ rad/s, and $k=1.5\times10^4$ N/m. For the same angle of incidence and incident ball spin, the exit ball spin can be increased to 216 rad/s with a 1.0 g cart and to 223 rad/s with a 0.1 g cart. A measure of the effectiveness of the cart in enhancing the ball spin is the tangential coefficient of restitution $e_x=-(v_{x2}-r\omega_2)/(v_{x1}-r\omega_1)$. When the ball is incident on the 10 g cart at $\theta_1=60°$ and exits at 162.8 rad/s, it does so with $e_x=0.62$. For the 0.1 g cart, $e_x=0.99$ at $\omega_2=223$ rad/s. The friction force created by motion of the cart acts to reduce v_{x2} while it increases ω_2, so a ball that bounces with increased spin bounces at an angle closer to the normal.

IV. DISCUSSION AND CONCLUSION

Tennis players apply topspin to a ball by swinging the racquet upward to meet the incoming ball, while simultaneously swinging the racquet forward. The ball slides across the strings for a short period and then grips the strings. In the absence of tangential motion of the strings, there would be no further change in ball spin after the sliding stage terminates, apart from effects due to tangential distortion of the ball. The latter effect has not been considered in this paper, but experimental data show that the tangential distortion of the ball can act to enhance the ball spin when the ball bounces off a rigid surface.[7,8] Our results show that additional spin can be expected as a result of the tangential motion of the strings within the string plane, provided the strings are allowed to return to their original position during the impact with the ball.

An enhancement of the present model would be to include an additional friction force acting on the cart to simulate the friction force that acts between overlapping strings. If this friction force is large enough, then the cart will not return to its original position when the impact is over, and the result will be a decrease in the outgoing spin of the ball. The latter result was included in our model by considering a heavy cart. Another enhancement would be to replace the single cart with a chain of masses and springs to simulate the flexible surface.

In the late 1970s, it was found that a large amount of spin could be imparted to a tennis ball if the strings were not woven. The International Tennis Federation banned the use of such stringing methods, insisting that the strings must be woven. Nevertheless, it is clear that “slippery” strings move relatively freely past each other even when they are woven, a result that is easy to demonstrate simply by pulling the strings aside by hand and then releasing them. Anecdotal evidence suggests that the result is an enhancement in the spin of the ball. It is probably for that reason that most professional tennis players now use polyester tennis strings rather than nylon or natural gut. The mechanism outlined in this paper shows, in a simplified manner, how the additional spin can be generated, and that even very light strings can contribute significantly to the spin imparted to the ball. The total mass of the strings in a racquet is typically about 15 g, and only a small fraction of that mass moves sideways when the ball impacts the strings. A small decrease in string mass, using thinner strings, may help increase the outgoing ball spin slightly, although a large decrease in string mass would be impractical because the strings would break too easily.

It is possible to enhance the spin of the ball in other sports, such as golf and baseball, by coating the striking implement with a suitably flexible material. Such an effect would be of considerable practical interest because the flight distance of a ball through the air is strongly enhanced when the ball spins backward.

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