Senior astrophysics Lab 4: Roche lobes

Name:

Checkpoints due: Friday 27 April 2018

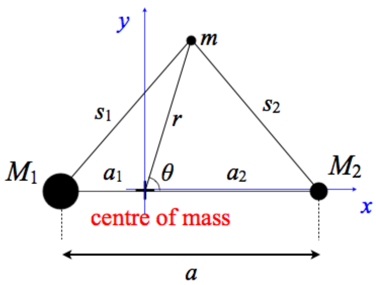
In this lab, you are exploring the gravitational potential around a binary star system. At various points, you are asked to sketch this potential, which could strain your artistic skills. You might like to save copies of the actual figures for future reference.

1 Introduction

In lectures we derived an expression for the effective gravitational potential Φ:

$$\Phi = -G\left(\frac{M_1}{s_1} + \frac{M_2}{s_2}\right) - \frac{1}{2}\omega^2 r^2 \qquad (1)$$

where s_1 and s_2 are the distance from stars 1 and 2.



We can write this in a more useful form if we set our origin at star 1, so

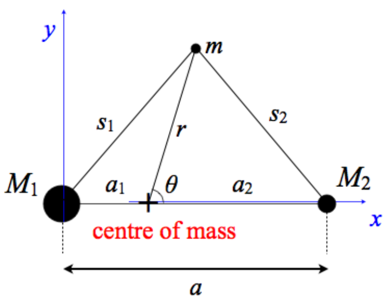
$$s_1^2 = x^2 + y^2 + z^2$$

$$s_2^2 = (x-a)^2 + y^2 + z^2$$

and the position of the centre of mass x_c is given by

$$\frac{x_c}{a} = \frac{M_2}{M_1 + M_2} = \frac{q}{1+q}$$

where $q \equiv M_2/M_1 \leq 1$ and the separation of the stars is a.



Then we have

$$\Phi = -G\left(\frac{M_1}{s_1} + \frac{M_2}{s_2}\right) - \frac{\omega^2}{2}\left[(x - x_c)^2 + y^2\right]$$

Even more useful is to write the *dimensionless Roche potential* Φ_N, by factoring out $-\omega^2/2 = -GM/2a^3$, and substituting $x \to x/a$ etc; then we get (check this!)

$$\Phi_N(x, y, z) = \frac{2}{(1+q)}\frac{1}{s_1} + \frac{2q}{(1+q)}\frac{1}{s_2} + \left(x - \frac{q}{(1+q)}\right)^2 + y^2 \quad (2)$$

Because we have scaled by a, the two stars are now at $x = 0$ and $x = 1$. Equation 2 describes the potential at any point r.

This single-parameter equation describes the shape of the potential surfaces, independent of the mass and size of the system. In this lab, we will be investigating some of the properties of this equation, using Matlab.

Note that we will only be exploring the potential in the x, y plane (the plane containing the binary orbit). The potential is defined in three dimensions, but we are most interested in how it varies in the orbital plane. Thus we will ignore z in what follows.

2 Exercise 1: Cross-sections along the y-axis

Q1 Create a cross-section of the gravitational potential Φ_N along the x-axis for a mass ratio $q = 0.2$. Draw the curve for x-values between -2.5 and 2.5, and show the location of the two stars. Notice that you will have to plot $-\Phi$, and set $y = 0$. Recall that $s_1 = \sqrt{x^2 + y^2}$ and is always positive, unlike x; and because we are restricting our analysis to the binary plane, $z = 0$.

MATLAB HINT: The first function has been written for you (see `phi_x.m` on eLearning). We have created a vector for the independent variable x in the range $[-2.5, 2.5]$, and set $y = 0$ to look at ϕ along the x-axis.
Copy this code into your working area and plot these values against x:

```
plot(x,-phi_x)
xlabel('x')
ylabel('\Phi')
axis([-2.5 2.5 -20 0])
```

Q2 Zoom in on your plot so you can see the region between the two stars clearly. What is the value for the potential at the L1 point? Give your answer to 4 significant figures.

Q3 Equation 2 has three additive terms. What do those three terms represent?

Q4 Plot each term separately on top of the curve you made in Q1 (as dashed lines).

Q5 Briefly explain the shape of each line. Where is the maximum (or minimum) for each curve, and why?

▶ **CP1**

Tutor's initials

3 Exercise 2: Contour plots

Q1 Now we want to draw contours of the potential in the x, y plane.

MATLAB HINT: You will need to create two vectors x and y, then use the function `meshgrid` to turn x and y into a grid

```
x=-2.5:0.01:2.5;
y=-2.5:0.01:2.5;
[Mx My] = meshgrid(x,y);
```

Define Φ_N on this grid, using equation 2. Copy your expression for Φ from Exercise 1, and replace every occurrence of x with Mx and every y with My:

```
phi_N = 2 ./ (((Mx.^2 + My.^2).^0.5)*(1+q)) + ...
```

etc.: make sure you're not missing any dots! Now draw contours for the same mass ratio you used in Ex. 1, using the Matlab function

```
contour(Mx, My, -phi_N, levels)
```

where `levels` is a vector of length n containing the n contour levels to be plotted. Try the values

```
levels=[-3.3 xxx -5 -15 -35]
contour(Mx, My, -phi_N, levels)
```

where xxx is a contour at the value determined in Q2 of Exercise 1.

Q2 Now make a surface plot (a "rubber-sheet diagram") using the spatial x and y coordinates, plus the value of the potential on the z-axis.

```
surf(Mx, My, phi)
colorbar
zlim([-50 0])          % set \Phi limits
shading interp         % remove grid lines (otherwise it's completely black!)
caxis([-50 0])         % set color axis to be the same as \Phi
```

Experiment with a few different views, using the *Camera* toolbar. Sketch and describe what you find. Indicate the five Lagrange points.

Q3 For each of the five Lagrange points, indicate whether they are a local maximum, local minimum, or saddle point. What does this indicate about their stability? Which points are the most stable? the most unstable?

▶ **CP2** Tutor's initials

4 Exercise 3: Shape of the potential

Q1 What happens to the shape of the potential if you change the mass ratio? You have drawn contours for $q = M_2/M_1 = 0.2$; what happens if $q = 1$? $q = 0.01$?

Q2 In writing Φ_N, you normalised all dimensions by the orbital separation a. What does this tell you about what happens to the shape of the potential if you keep the masses of your stars the same but change the distance between them?

▶ **CP3**

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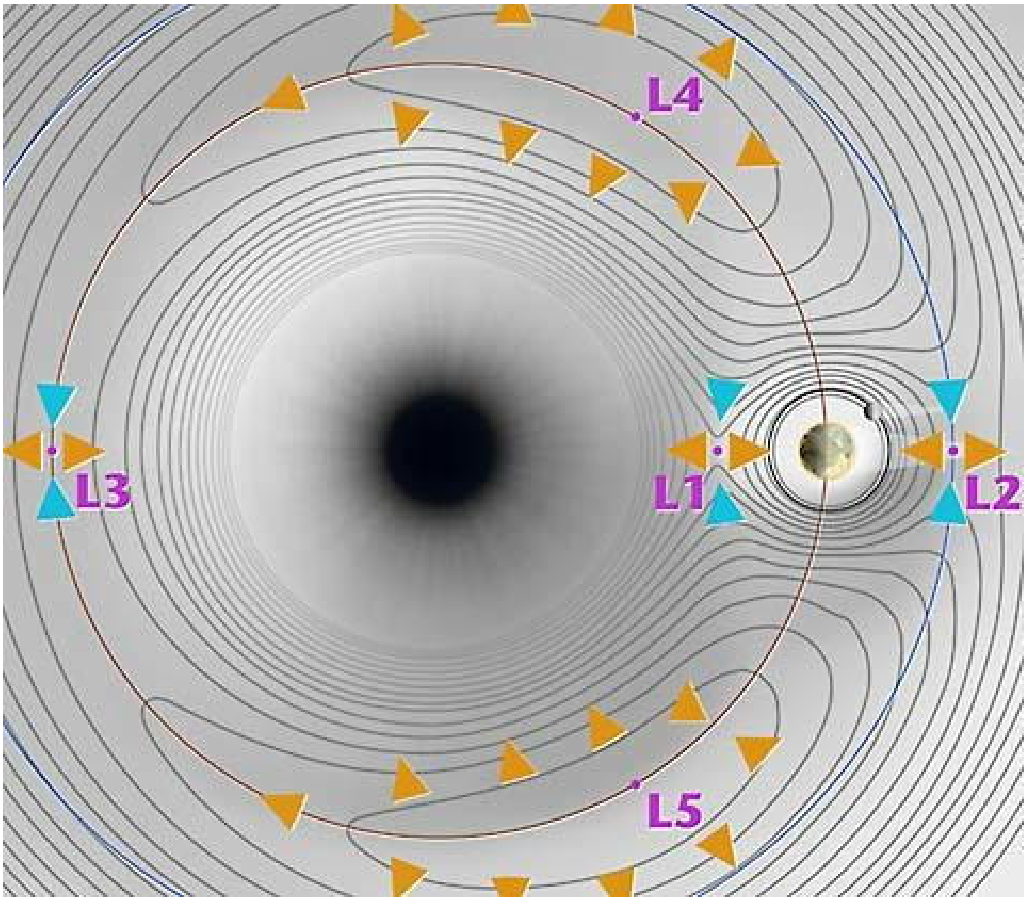
5 Discussion

According to your plot, it appears that all five Lagrange points are unstable.



from WMAP: The Lagrange points
(http://map.gsfc.nasa.gov/mission/observatory_l2.html)

In fact, the L4 and L5 points are *stable*, for a rather subtle reason: the potential in our rotating reference frame depends on *velocity* as well as position. When a test particle is placed at the L4 (or L5) point, it will (under small perturbations) tend to slide down the potential. However, as it does so, it gains speed, and the Coriolis force sends it into a stable orbit around the Lagrange point. This stability is only achieved if $M_1/M_2 > 24.96$ ($q < 0.04$), so the L4 and L5 points are stable for e.g. planets around the Sun (like the Jupiter Trojan satellites, where $M_\odot/M_J = 1047$), but will *not* in general be stable for binary stars.[1]

[1] A proper derivation of this stability can be found at https://map.gsfc.nasa.gov/ContentMedia/lagrange.pdf.