Igloo

An igloo is a hemispherical enclosure built of ice. Elmo's igloo has an inner radius of 2.55 m and the thickness of the ice is 0.30 m. This thickness can be considered small compared to the radius. Heat leaks out of the igloo at a rate determined by the thermal conductivity of ice, $k_{\text{ice}} = 1.67\ \text{W.m}^{-1}.\text{K}^{-1}$.

At what rate must thermal energy be generated inside the igloo to maintain a steady air temperature inside the igloo at 6.5 °C when the outside temperature is –40 °C?
Ignore all thermal energy losses by conduction through the ground, or any heat transfer by radiation or convection or leaks.

Solution

The igloo loses heat through its hemispherical surface, at a rate

$$H = \frac{dQ}{dt} = kA\frac{\Delta T}{\Delta x}$$

where $\Delta x = 0.30$ m is the thickness of the ice.

Inside radius of the hemisphere is $r_{\text{in}} = 2.55$ m, outside radius is $r_{\text{out}} = 2.85$ m, so take $r = r_{\text{av}} = 2.70$ m. Then the total surface area of the igloo is half the surface area of a sphere of radius r:

$$A = \tfrac{1}{2}\times 4\pi r^2 = (2\pi)(2.70)^2 = 45.80\ \text{m}^2$$

The temperature difference between inside and out is $\Delta T = -40 - 6.5 = -46.50$ °C.
Thus the heat loss by conduction is

$$H = (1.67)(45.80)\frac{-46.50}{0.30} = -1.2\times 10^4\ \text{W}$$

where the minus sign reminds us that heat is flowing *out*.

In order to maintain a steady temperature, the rate of energy production must equal this rate of heat loss, so Elmo needs to produce 12 kW of power inside the igloo to maintain thermal equilibrium.