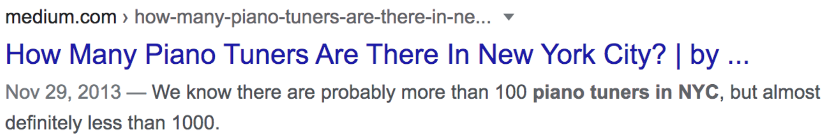
PHYS 2013/2913/2923 Tutorial 2

Here we look at how astronomers estimate. We don't care how big the question, how approximate the answers, if it gives us a better understanding. Order of magnitude approximations can be made for anything, and it's how astronomers start any calculation.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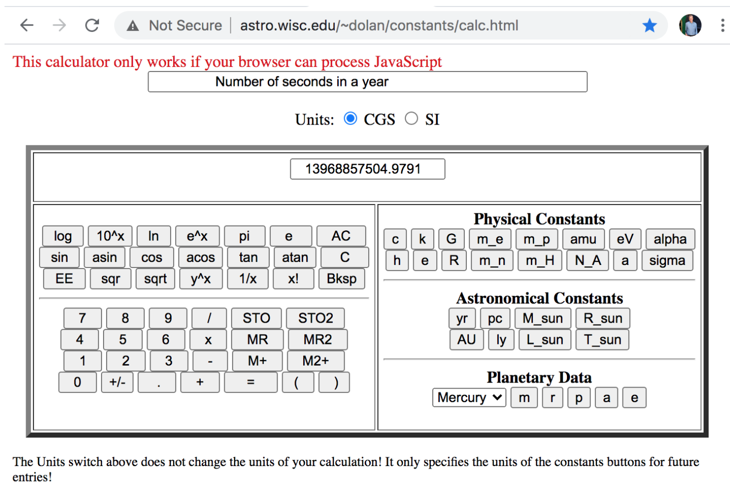
How Many Piano Tuners Are There In New York City? | by ...

Nov 29, 2013 — We know there are probably more than 100 **piano tuners in NYC**, but almost definitely less than 1000.

1. I find the online calculator at the link below really useful; it works with SI or cgs units and has fundamental constants for astronomy in both units.

In class, using the calculator, I would like you to work out what is $1/H_0$, where Hubble's parameter $H_0 = 70$ km s^{-1} Mpc^{-1}. What are the units? What is the significance of this quantity?



List of Ingredients		
photons:	$\Omega_{\gamma,0} = 5.0 \times 10^{-5}$	
neutrinos:	$\Omega_{\nu,0} = 3.4 \times 10^{-5}$	
total radiation:	$\Omega_{r,0} = 8.4 \times 10^{-5}$	
baryonic matter:	$\Omega_{\text{bary},0} = 0.04$	
nonbaryonic dark matter:	$\Omega_{\text{dm},0} = 0.26$	
total matter:	$\Omega_{m,0} = 0.30$	
cosmological constant:	$\Omega_{\Lambda,0} \approx 0.70$	

Important Epochs		
radiation-matter equality:	$a_{rm} = 2.8 \times 10^{-4}$	$t_{rm} = 4.7 \times 10^4\,\text{yr}$
matter-lambda equality:	$a_{m\Lambda} = 0.75$	$t_{m\Lambda} = 9.8\,\text{Gyr}$
Now:	$a_0 = 1$	$t_0 = 13.5\,\text{Gyr}$

2. According to a geophysicist colleague at the University of Hawaii, there are 10^{20} grains of sand on Earth. How does that compare with the number of stars or planets in the observable Universe?

Approach: For all calculations that follow, the Universe is considered to be flat ($k = 0$, $\kappa = 0$, or whatever convention is used for curvature; this simplifies the algebra a lot.) Compute the total volume of the Universe out to the observable horizon. What's the appropriate distance metric? From what we observe today, compute the total number of baryons and photons, and assume true over the same volume. Now compute the total number of stars and total number of galaxies.

(If time allows, I will show solutions to the integrals using my free Wolfram Cloud `Mathematica` account.)

Reminder: proper distance from integrating over the radial comoving coordinate r,

$$d_P(t) = a(t) \int_0^r dr = a(t) r \tag{1}$$

Remember how we bring in Hubble's parameter:

$$\dot{d}_P = \dot{a} r = \frac{\dot{a}}{a} d_P \tag{2}$$

We can evaluate today ($t = t_0$):

$$v_P(t_0) = H_0\, d_P(t_0) \tag{3}$$

The most distant object you can see is one for which the light emitted at $t = 0$ is just now reaching you at $t = t_0$. From (1) and (3), the Hubble distance is one measure of the size of the observable Universe (radiation dominated):

$$\begin{aligned} d_H(t_0) &= c\int_0^{t_0} \frac{dt}{a(t)} && (4)\\ &= 2ct_0 && (5)\\ &= c/H_0 && (6)\\ &= 4300 \text{ Mpc} && (7) \end{aligned}$$

where the age of the Universe is $t_0 = 1/2H_0$. There are galaxies beyond this horizon travelling faster than c with respect to us, but no information is being passed.

From (1) and (3), the Hubble distance is one measure of the size of the observable Universe (matter dominated):

$$\begin{aligned} d_H(t_0) &= c\int_0^{t_0} \frac{dt}{a(t)} && (8)\\ &= 3ct_0 && (9)\\ &= 2c/H_0 && (10)\\ &= 8600 \text{ Mpc} && (11) \end{aligned}$$

where the age of the Universe is $t_0 = 2/3H_0$ (Einstein-de Sitter Universe).

If we account for Λ acceleration, where $a(t) = e^{H_0(t-t_0)}$,

$$\begin{aligned} d_H(t_0) &= c\int_0^{t_0} \frac{dt}{a(t)} && (12)\\ &= 3.20\, c/H_0 && (13)\\ &= 14000 \text{ Mpc} && (14) \end{aligned}$$

where the age of the Universe is $t_0 \approx H_0^{-1}$ (de Sitter Universe).

The proper volume inside the horizon,

$$\begin{aligned} V_H &= \frac{4\pi}{3} d_H^3 && (15)\\ &= 137(c/H_0)^3 && (16)\\ &= 1.15 \times 10^{13} \text{ Mpc}^3 && (17) \end{aligned}$$

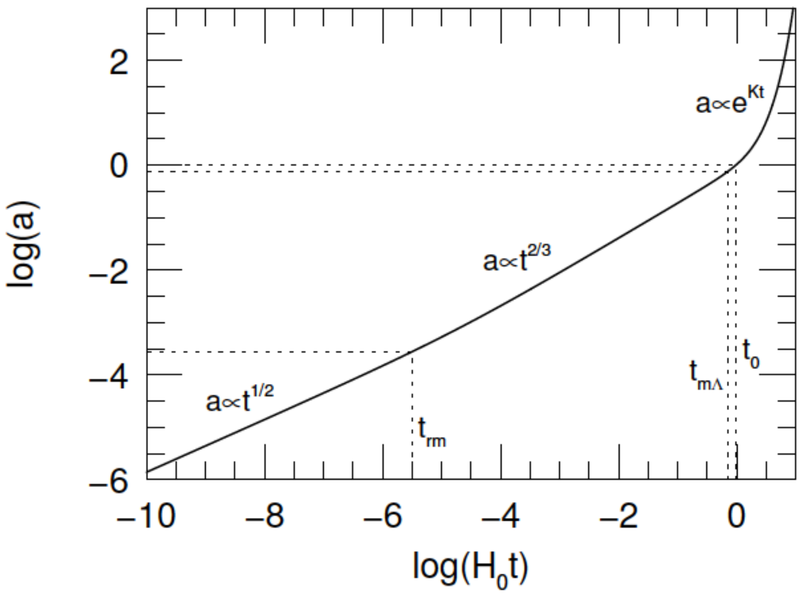
The mass density

$$\begin{aligned} \rho_{m,0} &= \Omega_{m,0}\, \rho_{c,0} && (18)\\ &= 0.3(1.28 \times 10^{11} \text{ M}_\odot \text{ Mpc}^{-3}) && (19)\\ &= 4 \times 10^{10} \text{ M}_\odot \text{ Mpc}^{-3} && (20) \end{aligned}$$

Thus the total mass within the horizon is

$$\begin{aligned} M_H &= \rho_{m,0} V_H &(21)\\ &= 4.6 \times 10^{23}\ \mathrm{M}_\odot &(22)\end{aligned}$$

with 90% in dark matter, 10% in stars and gas. So how do stars and planets compare to grains of sand?



3. In the plot, the Universe has interesting transition epochs where t_0 is the present. When did the radiation-dominated Universe give way to the matter-dominated Universe at $t = t_{\rm rm}$?

Approach: work in terms of scale factor a, cosmic time t, and redshift z. All of these can be easily related for a given cosmology (Universe). At $t = t_{\rm rm}$, using the Friedmann equation:

Even though the energy **density** declines, the total radiation and matter content (number of particles) is roughly constant over cosmic time.

$$\frac{H^2}{H_0^2} = \frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} \tag{23}$$

Before that time, the first term dominated; after that time, the second term dominated. All "0" subscripts refer to the present. This is a flat Universe dominated by radiation and matter (emergent dark energy comes later).

Another version of that formula:

$$H(t)^2 = \frac{8\pi G}{3c^2}(\varepsilon_r(t) + \varepsilon_m(t)) \tag{24}$$

Today's ratio

$$\frac{\Omega_{r,0}}{\Omega_{m,0}} = \frac{\varepsilon_{r,0}}{\varepsilon_{m,0}} \approx 2.8 \times 10^{-4} \tag{25}$$

But note that the following is also true:

$$\frac{\varepsilon_r(a)}{\varepsilon_m(a)} = \frac{\varepsilon_{r,0}/a^4}{\varepsilon_{m,0}/a^3} \tag{26}$$

Scale factor a and time t are useful interchangeable variables; just set $a = a_{\rm rm}$ in the past when $\varepsilon_r = \varepsilon_m$, so that

$$a_{\rm rm} = \frac{\varepsilon_{r,0}}{\varepsilon_{m,0}} \approx 2.8 \times 10^{-4} \tag{27}$$

Remember always this important formula:

$$z = \frac{a(t_0)}{a(t_e)} - 1 \tag{28}$$

that works for all cosmologies in this course; t_e is when the photon was emitted and $a(t_0) = 1$ by definition. So that:

$$a = \frac{1}{1+z} \tag{29}$$

This tells us that

$$a_{\rm rm} = \frac{1}{1+z_{\rm rm}} \tag{30}$$

and $z_{\rm rm} \approx 3600$. So redshifts and scale factors are easy - how about actual cosmic times? It's worth seeing this derived at least once to understand how it all follows.

So you've seen at various times already that $a \propto t^{1/2}$ during the radiation epoch ($a \ll a_{\rm rm}$), $a \propto t^{2/3}$ during the matter epoch ($a \gg a_{\rm rm}$). Let's find a formula for t where this is true.

Remembering $H(t) = \frac{\dot{a}}{a}$, it's easy enough to rearrange formula (23) (do that in your own time) to get:

$$H_0\, dt = \frac{a}{\sqrt{\Omega_{r,0}}}(1 + \frac{a}{a_{\rm rm}})^{-1/2}\, da \tag{31}$$

We can integrate this formula by hand or with a symbolic manipulator like `Mathematica` to get:

$$H_0\, t = \frac{4a_{\rm rm}^2}{3\sqrt{\Omega_{r,0}}}(1 - (1 - \frac{a}{2a_{\rm rm}})(1 + \frac{a}{a_{\rm rm}})^{1/2}) \tag{32}$$

In the limit of $a \ll a_{\rm rm}$,

$$a(t) \approx (2\sqrt{\Omega_{r,0}}H_0 t)^{1/2} \tag{33}$$

In the limit of $a \gg a_{\rm rm}$,

$$a(t) \approx (\frac{3}{2}\sqrt{\Omega_{m,0}}H_0 t)^{2/3} \tag{34}$$

And now to answer the question, $a = a_{\rm rm}$ so that

$$\begin{aligned} t_{\rm rm} &= \frac{4}{3}(1-\frac{1}{\sqrt{2}})\frac{a_{\rm rm}^2}{\sqrt{\Omega_{r,0}}}H_0^{-1} \approx 0.39\frac{\Omega_{r,0}^{3/2}}{\Omega_{m,0}^2}H_0^{-1} && (35)\\ &= 3.3\times 10^{-6}H_0^{-1} && (36)\\ &= 47,000 \text{ years} && (37)\end{aligned}$$

4. When did the matter-dominated Universe give way to the Λ-dominated universe at $t = t_{\rm m\Lambda}$?

The solution follows above very closely where we recognize that $\Omega_{r,0}$ can be ignored because it is so small compared to matter and Λ dominance today. Thus

$$a_{m\Lambda} = (\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}})^{1/3} = (\frac{\Omega_{m,0}}{1-\Omega_{m,0}})^{1/3} \quad (38)$$

Once again, equation (1) can be integrated to give

$$H_0\, t = \frac{2}{3\sqrt{1-\Omega_{m,0}}}\ln\left[(a/a_{m\Lambda})^{3/2} + \sqrt{1+(a/a_{m\Lambda})^3}\right] \quad (39)$$

In the limit of $a \ll a_{\rm m\Lambda}$,

$$a(t) \approx (\frac{3}{2}\sqrt{\Omega_{m,0}}H_0 t)^{2/3} \quad (40)$$

In the limit of $a \gg a_{\rm m\Lambda}$,

$$a(t) \approx a_{\rm m\Lambda}\exp(\sqrt{1-\Omega_{m,0}}H_0 t) \quad (41)$$

This has the correct form for a flat, Λ-dominated Universe, much like the exponential dependence of inflation.

And now to answer the question, $a = a_{\rm m\Lambda}$ so that

$$\begin{aligned} t_{m\Lambda} &= \frac{2H_0^{-1}}{3\sqrt{1-\Omega_{m,0}}}\ln\left[1+\sqrt{2}\right] && (42)\\ &= 0.702H_0^{-1} && (43)\\ &= 9.8 \text{ Gyr} && (44)\end{aligned}$$

where $H_0 = 70$ km s^{-1} Mpc^{-1}. So this all started to happen about the time the Solar System formed. No connection!

5. What is the age of the Universe today?

If we plug (16) into (17), we get a general formula for the age today:

$$t_0 = \frac{2H_0^{-1}}{3\sqrt{1-\Omega_{m,0}}} \ln\left[\frac{\sqrt{1-\Omega_{m,0}}+1}{\sqrt{\Omega_{m,0}}}\right] \quad (45)$$

$$= 0.964 H_0^{-1} \quad (46)$$

$$= 13.6 \text{ Gyr} \quad (47)$$

Note that $H_0^{-1} = 14.0$ Gyr is the age of the Universe to within a few percent; the actual age from the WMAP and Planck CMB satellites is 13.8 Gyr.