PHYS377 – ASTROPHYSICS I

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Weeks 1-6

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Contents

REFERENCES

These lecture notes are based on lectures written by I.L. Guy. In addition, the following sources have been used:

1. Bowers R. and Deeming T., *Astrophysics*, Jones and Bartlett, 1984 (QB461.B64)

   Broad coverage in two volumes. Chapter 5 of the first volume contains material related to basic radiation and energy transfer (chapter 5).

2. Griffiths, D.J., *Introduction to Electrodynamics*, Prentice-Hall, 1989

   Treatment of electrodynamics at a more discursive level than Jackson.

3. Jackson J.D., *Classical Electrodynamics*, Wiley, 1975 (QC631.J3)

   One of the standard texts on electromagnetism. Contains material relevant to the sections on radiation.

4. Rybicki G.B. and Lightman A. P., *Radiative Processes in Astrophysics*, John Wiley and Sons, 1979 (QB461.R88)

   Covers the radiation aspects of the course very thoroughly. Contains problems with worked solutions.

Chapter 1

Properties of Radiation

References: Rybicki and Lightman, Bowers and Deeming

1.1 Introduction

Almost all of the information we receive from astrophysical objects comes in the form of electromagnetic radiation. Hence we begin this course by defining some terms used to describe macroscopic properties of radiation. Mechanisms for producing radiation are not considered at this point.

The *solid angle* subtended at a point by a projected area dA a distance R from the point is $d\Omega = dA/R^2$. In spherical polar co-ordinates $d\Omega = \sin\theta\, d\theta\, d\phi$. The unit of solid angle is the steradian. The complete solid angle (including all directions) is 4π steradians, since $\int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi = 4\pi$.

In astrophysics texts, cgs units are frequently met. For completeness, note that an erg is 10^{-7} Joules and a dyne is 10^{-5} Newtons.

As a review from earlier physics courses a list of the names, wavelengths, energies and temperatures of various ranges of the electromagnetic spectrum is tabulated below.

	λ (m)	ν (Hz)	E (eV)	T (K)
radio	$> 10^{-3}$	$< 10^{12}$	$< 10^{-3}$	$< 10^{1}$
IR	$10^{-6} \rightarrow 10^{-3}$	$10^{12} \rightarrow 10^{15}$	$10^{-3} \rightarrow 10^{0}$	10^{4}
visible	10^{-6}	10^{15}	10^{0}	$10^{4.5}$
UV	$10^{-3} \rightarrow 10^{-6}$	$10^{15} \rightarrow 10^{17}$	$10^{0} \rightarrow 10^{2}$	$10^{5} \rightarrow 10^{6}$
X ray	$10^{-11} \rightarrow 10^{-8}$	$10^{17} \rightarrow 10^{19}$	$10^{2} \rightarrow 10^{5}$	$10^{6} \rightarrow 10^{9}$
γ ray	$< 10^{-11}$	$> 10^{19}$	$> 10^{5}$	$>10^{9}$

1.2 Terms used to describe radiation

When the scale of a system greatly exceeds the wavelength of radiation the radiation can be considered to travel in straight lines (as rays) in free space or in a homogenous medium. The resulting theory of radiation is known as transfer theory, and is widely applicable in astrophysics.

1.2.1 Radiative flux

Consider a small element of area dA exposed to radiation for a time dt. The total energy dE passing through the surface in that time is expected to be proportional to $dA\,dt$:

$$dE = F\,dA\,dt. \tag{1.1}$$

The constant of proportionality F is the radiative flux, i.e. energy passing through unit area in unit time. The dimensions of flux are $\mathrm{Jm^{-2}s^{-1}}$, i.e. $\mathrm{Wm^{-2}}$.

Frequently the flux per unit range of frequency is specified. The above definition then becomes

$$dE = F_\nu\,dA\,dt\,d\nu. \tag{1.2}$$

1.2.2 Intensity or brightness

The radiation flux may include incoming rays from all directions. The intensity (or specific intensity) gives the energy passing through unit area in unit time from a specified direction. Because a single ray carries essentially no energy, it is necessary to consider incoming rays from a small range of directions defined by a solid angle $d\Omega$. The formal definition of intensity is as follows. Consider a small area dA perpendicular to a given ray defined by a direction $\mathbf{\Omega}$ and a small solid angle $d\Omega$ about the ray. The total energy crossing dA from all rays within the solid angle $d\Omega$ is

$$dE = I_\nu\,dA\,dt\,d\nu\,d\Omega. \tag{1.3}$$

The quantity $I_\nu = I_\nu(\nu, \mathbf{\Omega})$ is the intensity, with units of $\mathrm{Wm^{-2}Hz^{-1}ster^{-1}}$.

Next consider the energy crossing dA from rays in a small solid angle about a direction $\mathbf{\Omega}$ at an angle θ to the normal to dA. The area pependicular to the direction $\mathbf{\Omega}$ is $\cos\theta\,dA$, and so in this case

$$dE = I_\nu\,\cos\theta\,dA\,dt\,d\nu\,d\Omega. \tag{1.4}$$

The total energy associated with all directions is obtained by integrating (1.4) over solid angle. Comparing the result with the definition of flux, Equation (1.2) gives the relation between flux and intensity:

$$F_\nu = \int I_\nu \cos\theta\,d\Omega. \tag{1.5}$$

The term *brightness* (denoted B_ν) describes the same physical quantity as intensity. As a general rule, 'brightness' is used when describing radiation at a source, while 'intensity' is used when talking about radiation received by an observer.

It is also common to refer to average intensity J_ν, which is I_ν averaged over all directions:

$$J_\nu = \frac{1}{4\pi}\int I_\nu d\Omega. \tag{1.6}$$

An important property of specific intensity is that it is conserved along a ray in free space. To see this, consider a chosen ray and choose two points along the ray separated by a distance R. Construct small areas dA_1 and dA_2 normal to the ray at the chosen points. In free space the amount of energy carried by rays intersecting both dA_1 and dA_2 must be constant, and so we have

$$I_{\nu,1}\,dA_1 dt d\nu d\Omega_1 = I_{\nu,2}\,dA_2 dt d\nu d\Omega_2, \tag{1.7}$$

where $I_{\nu,1}$ and $I_{\nu,2}$ are the specific intensities at the two chosen points, and $d\Omega_1$ and $d\Omega_2$ are the solid angles subtended at each point by the small area at the other point. Since $d\Omega_1 = dA_2/R^2$ and $d\Omega_2 = dA_1/R^2$, we have

$$I_{\nu,1} = I_{\nu,2}, \tag{1.8}$$

and since the distance between the points is arbitrary this means

$$\frac{dI_\nu}{ds} = 0 \tag{1.9}$$

where s is distance along a ray. In other words specific intensity is constant along a ray.

1.2.3 Momentum flux

We can think of the integrand in (1.5) as defining the differential amount of flux $dF_\nu = I_\nu \cos\theta d\Omega$ associated with the solid angle $d\Omega$, which is oriented at an angle θ to an area dA. The momentum of a photon with energy E is E/c, and so the momentum flux (momentum per unit time, per unit area, which is equivalent to pressure) associated with the radiation coming from the solid angle $d\Omega$ and crossing dA is dF_ν/c. The component of the momentum flux normal to dA is $dF_\nu \cos\theta/c$, and so the total momentum flux normal to dA is

$$p_\nu = \frac{1}{c}\int I_\nu \cos^2\theta \, d\Omega. \tag{1.10}$$

The units of momentum flux are $\mathrm{Nm^{-2}Hz^{-1}}$.

1.2.4 Total (integrated) quantities

It is always possible to integrate over frequency to obtain the total intensity, flux, and momentum flux:

$$I = \int_0^\infty I_\nu d\nu, \tag{1.11}$$

$$F = \int_0^\infty F_\nu d\nu, \tag{1.12}$$

$$p = \int_0^\infty p_\nu d\nu. \tag{1.13}$$

1.2.5 Energy density

The specific energy density u_ν is usually taken as the radiation energy per unit volume, per unit frequency range. To determine this we introduce the energy density $u_\nu(\mathbf{\Omega})$ per unit frequency and per unit solid angle associated with a direction $\mathbf{\Omega}$. (The units of $u_\nu(\mathbf{\Omega})$ are $\mathrm{Jm^{-3}Hz^{-1}ster^{-1}}$.) Consider a cylinder with axis along $\mathbf{\Omega}$ and with cross-sectional area dA and length $c\,dt$. The total energy in this cylinder associated with rays about $\mathbf{\Omega}$ is

$$dE = u_\nu(\mathbf{\Omega})\, dA\, cdt\, d\Omega\, d\nu. \tag{1.14}$$

All of the radiation moving in the direction $\mathbf{\Omega}$ will cross the area dA at the end of the cylinder in the time dt. Hence another estimate for the energy in the cylinder is provided by (1.3). Equating (1.3) and (1.14) gives

$$u_\nu(\mathbf{\Omega}) = \frac{I_\nu}{c}, \tag{1.15}$$

and we see that the specific intensity is essentially the specific energy density associated with the given direction. Integrating over all directions gives

$$u_\nu = \int u_\nu(\mathbf{\Omega})\, d\Omega = \frac{1}{c}\int I_\nu \, d\Omega, \tag{1.16}$$

or

$$u_\nu = \frac{4\pi}{c} J_\nu, \tag{1.17}$$

in terms of the mean intensity J_ν. The total radiation energy density u (in Jm^{-3}) is obtained by integrating over all frequencies:

$$u = \frac{4\pi}{c} \int_0^\infty J_\nu \, d\nu. \tag{1.18}$$

In the tutorial problems for this week it is shown that for isotropic radiation (I_ν is independent of direction) there is a useful relation between the total energy density and total radiation pressure:

$$p = \frac{1}{3} u. \tag{1.19}$$

1.2.6 Inverse square law

How is the constancy of specific intensity along a ray in free space consistent with the inverse square law for decline in flux with distance from a source? To understand this point, consider Figure 1.1. A sphere with radius R emits with a uniform brightness B_0 (all rays leaving the sphere have the intensity B_0). The total flux at a point P at a distance r from the centre of the sphere is

$$\begin{aligned} F &= \int I \cos\theta \, d\Omega \\ &= B_0 \int_0^{2\pi} d\phi \int_0^{\theta_c} \sin\theta \cos\theta \, d\theta, \end{aligned} \tag{1.20}$$

where $\theta_c = \sin^{-1} R/r$ is the angle of a ray from P that is tangent to the sphere, as shown in Figure 1.1. Performing the integrals gives

$$F = \pi B_0 \left(\frac{R}{r}\right)^2, \tag{1.21}$$

which is the inverse square law. If we substitute $r = R$ we obtain the result that the flux at a surface of uniform brightness B_0 is

$$F = \pi B_0. \tag{1.22}$$

1.3 Thermal radiation

In an enclosure with walls maintained at a constant temperature T, radiation reaches a state of thermodynamic equilibrium with the enclosure, i.e. there is a state of balance between emission and absorption of radiation by the walls of the enclosure. The intensity of the radiation field inside the enclosure is then described by the Planck law,

$$B_\nu = \frac{2h\nu^3}{c^2} \left[\exp\left(\frac{h\nu}{k_B T}\right) - 1\right]^{-1}. \tag{1.23}$$

The same spectrum is produced by a hypothetical 'blackbody,' i.e. a body that absorbs (and does not reflect) incident radiation.

The quantity B_ν is the brightness per unit frequency range. The brightness per unit wavelength range, B_λ is obtained by noting that $B_\lambda = B_\nu \, |d\nu/d\lambda|$, and hence

$$B_\lambda = \frac{2hc^2}{\lambda^5} \left[\exp\left(\frac{hc}{\lambda k_B T}\right) - 1\right]^{-1}. \tag{1.24}$$

Figure 1.2 shows the Planck spectrum for temperatures $T = 1, 10, 100, ..., 10^8$K (the lowest curve in the figure is the spectrum for $T = 1$K, and the highest curve is for $T = 10^8$K).

Two approximations to the Planck law are commonly encountered.

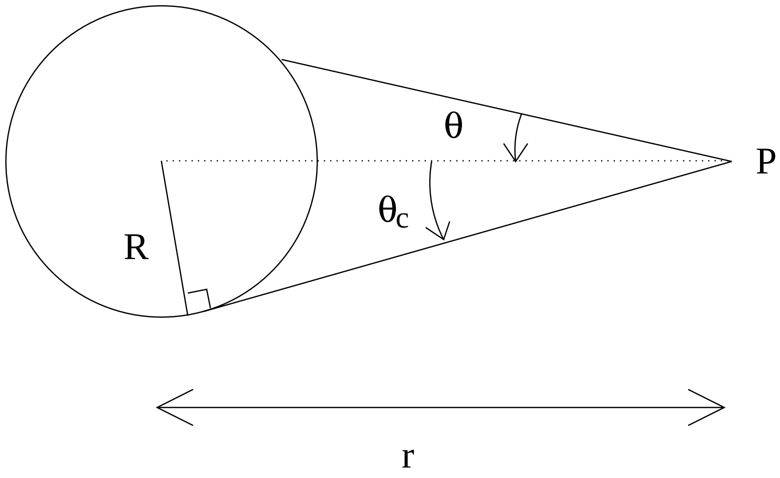


Figure 1.1: Flux from a sphere of uniform brightness.

1.3.1 The Rayleigh-Jeans law

If $h\nu \ll k_B T$ (i.e. at low frequencies and/or high T, situations frequently met in the radio spectrum) then the exponential in the denominator can be expanded as a series and terms above first order can be dropped. This leads to

$$B_\nu = \frac{2\nu^2}{c^2} k_B T. \tag{1.25}$$

Equation (1.25) is the Rayleigh-Jeans law, and corresponds to the power-law behaviour of the spectra at low frequencies in Figure 1.2.

1.3.2 The Wien law

If $h\nu \gg k_B T$ (i.e. high ν and/or low T), then the exponential term in the denominator is much larger than unity, and we have

$$B_\nu = \frac{2h\nu^3}{c^2} \exp\left(\frac{-h\nu}{k_B T}\right), \tag{1.26}$$

which is known as the Wien law. This behaviour corresponds to the steep decline in the spectra at large frequency seen in Figure 1.2.

There are two other useful relations for thermal radiation.

1.3.3 The Stefan-Boltzmann law

Integrating (1.23) over all frequencies leads to

$$B(T) = \int_0^\infty B_\nu \, d\nu = \frac{2k_B^4 T^4}{c^2 h^3} \int_0^\infty \frac{x^3}{e^x - 1} dx. \tag{1.27}$$

The integral is found in standard tables and has the value $\pi^4/15$. Hence we have

$$B(T) = \frac{\sigma}{\pi} T^4, \tag{1.28}$$

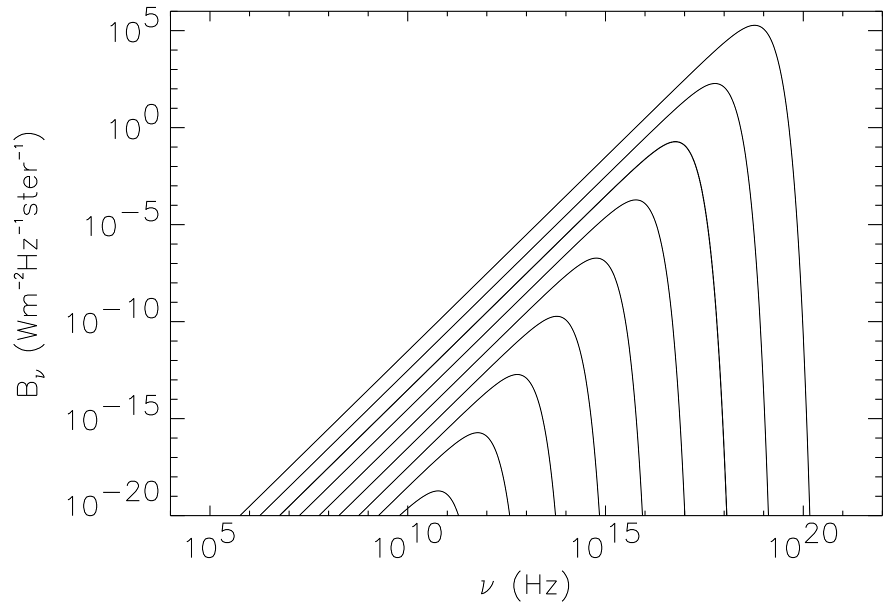


Figure 1.2: Blackbody spectra for $T = 1, 10, 100, ..., 10^8$K.

where

$$\sigma = \frac{2\pi^5 k_B^4}{15c^2h^3} \approx 5.67 \times 10^{-8}\mathrm{Wm}^{-2}\mathrm{K}^{-4} \tag{1.29}$$

is the Stefan-Boltzmann constant. Using the earlier result (1.22), we have that the emergent flux from a surface of uniform brightness is π times the brightness, and hence the flux at the surface of a blackbody is

$$F = \sigma T^4, \tag{1.30}$$

which is the usual form of the Stefan-Boltzmann law. The steep dependence of flux on temperature in this law is apparent in Figure 1.2: the flux is the area under each Planck curve, and this clearly increases rapidly with temperature.

1.3.4 The Wien displacement law

The Wien displacement law gives the value of the frequency, $\nu_{\max}$, at which the Planck spectrum is a maximum. To determine this we need to solve

$$\left.\frac{\partial B_\nu}{\partial \nu}\right|_{\nu=\nu_{\max}} = 0. \tag{1.31}$$

This leads to the equation $x = 3(1 - e^{-x})$, where $x = h\nu_{\max}/(k_B T)$, which has the (approximate) solution $x \approx 2.82$. Hence we have

$$h\nu_{\max} \approx 2.82 k_B T, \tag{1.32}$$

or

$$\nu_{\max} \approx 5.88 \times 10^{10} T \text{ Hz}. \tag{1.33}$$

The linear increase of $\nu_{\max}$ with temperature is apparent in Figure 1.2.

The wavelength $\lambda_{\rm max}$ at which B_λ is maximum can be determined by a similar procedure. The result is

$$\lambda_{\rm max} T \approx 0.00290 \text{ m K}. \tag{1.34}$$

Note that the peaks of B_ν and B_λ do not occur at the same places in frequency and wavelength, i.e. $\lambda_{\rm max}\nu_{\rm max} \neq c$.

1.3.5 Temperature from the Planck Law.

The radiation emitted by a body can be used to obtain a measure of its temperature, by matching it in some way to the Planck law. There are different ways of doing this, leading to different measures of temperature.

The Brightness Temperature

This is obtained by matching the measured I_ν with that from the Planck law (or one of its approximations), i.e. for any value of I_ν the brightness temperature $T_b(\nu)$ is defined by

$$I_\nu = B_\nu(T_b), \tag{1.35}$$

where the right hand side of (1.35) is the Planck spectrum. At radio frequencies it is common to use the Rayleigh-Jeans law, in which case

$$B_\nu = \frac{2\nu^2}{c^2} k_B T_b \tag{1.36}$$

and hence

$$T_b = \frac{c^2}{2\nu^2 k_B} I_\nu. \tag{1.37}$$

If the emitting object is a blackbody, then the brightness temperature will be the temperature of the source. Coherent emission processes (e.g. laser or maser emission, non-linear emission processes) may produce radiation with a brightness temperature that far exceeds the physical temperature of the source, and this can be used as an indicator of the emission mechamism.

Effective temperature

A second approach involves equating the flux at the source with that predicted by the Stefan-Boltzmann law for a blackbody. This leads to the effective temperature, $T_{\rm eff}$:

$$F = \sigma T_{\rm eff}^4. \tag{1.38}$$

If the emitting object is a blackbody, then the effective temperature will be the temperature of the source.

An example of effective temperature is provided by the Sun. The spectrum of the Sun roughly resembles a blackbody, with pieces missing due to absorption in the solar atmosphere. Modern spacecraft measurements give the flux of energy from the Sun at the Earth to be $F_{\rm E} = 1.368\,{\rm kW\,m^{-2}}$. The Earth-Sun distance (an astronomical unit, or AU) is $r_{\rm AU} = 1.496\times10^{11}$m, so the total radiant output of the Sun (the solar luminosity) is $L_\odot = 4\pi r_{\rm AU}^2 F = 3.85 \times 10^{26}\,{\rm W}$. The radius of the Sun is $R_\odot = 6.96 \times 10^8\,{\rm m}$, so the flux at the surface of the Sun is $F_\odot = L_\odot/(4\pi R_\odot^2) = 6.32 \times 10^7\,{\rm W\,m^{-2}}$. Equating this with $\sigma T_{\rm eff}^4$ gives an effective temperature $T_{\rm eff} = 5778\,{\rm K}$. By the Wien displacement law (1.34), the peak of the spectrum of the Sun is at about $5 \times 10^{-7}\,{\rm m}$, in the yellow part of the visible spectrum. Effective temperature (or equivalently colour) is used to categorise stars. For comparison, hot blue-white stars such as Sirius can have an effective temperature around $40,000\,{\rm K}$, whereas red giant stars such as Betelguese have $T_{\rm eff} \approx 3000\,{\rm K}$.

Colour Temperature

Often objects emit a spectrum of radiation similar to that of the Planck law for a blackbody. The colour temperature is obtained by fitting the frequency distribution of the observed object to that expected from a blackbody. This may be as simple as measuring the peak wavelength and using the Wien displacement law.

A good example of fitting to a blackbody spectrum is provided by the cosmic microwave background (CMB), the background radiation in the universe that is believed to be a remnant of the big bang, and hence to have cosmological significance. Figure 1.3 shows measurements of the CMB from a variety of sources, together with the fit to a blackbody at 2.726 K.

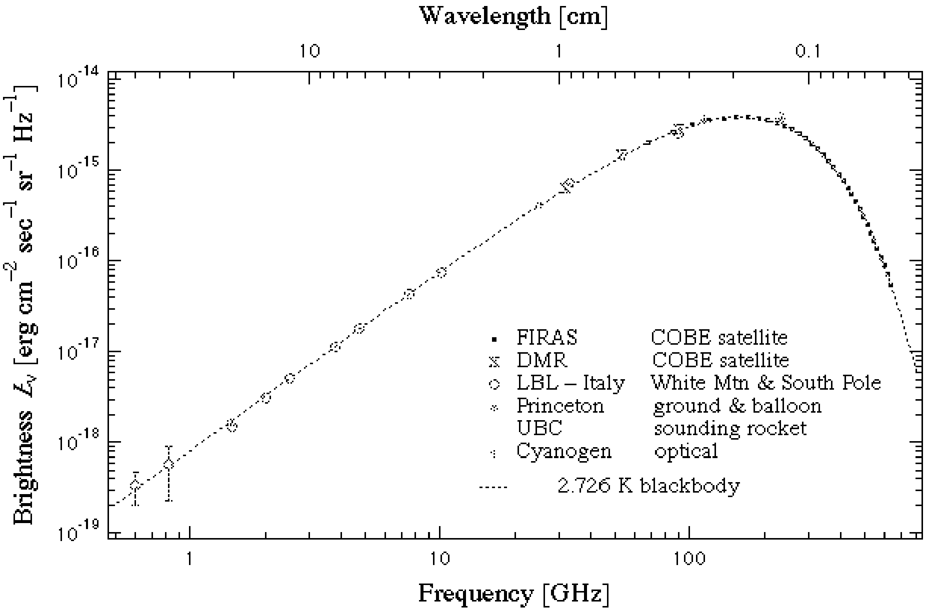


Figure 1.3: Cosmic microwave background measurements, and the fit to a blackbody spectrum. (From `http : //spectrum.lbl.gov/www/cobe/CMB_intensity.gif`.)

Problem Set 1

1. Consider an infinite plane with uniform brightness B_0. Show that the flux at a point at a distance r from the plane is $F = \pi B_0$, independent of the value of r.

2. What is the relation between I_ν and I_λ?

3. Show that for an isotropic radiation field (I_ν is independent of direction) the flux is zero, $I_\nu = J_\nu$, and the radiation pressure and energy density satisfy Equation (1.19).

4. The Sun radiates approximately as a blackbody at 5778 K. A perfectly black copper sphere is placed one astronomical unit from the Sun, at which distance the Sun's diameter subtends an angle of 0.5°. What is the equilibrium temperature of the copper sphere ?

   (To answer this question, use only the numerical values supplied in the question.)

Chapter 2

Radiative Transfer

References: Rybicki and Lightman, Bowers and Deeming

So far we have considered radiation in free space. If a ray passes through matter, energy may be added or subtracted from the ray by three processes: *emission*, *absorption* and *scattering*. The theory describing the resulting variation in the intensity of the radiation is radiative transfer. The approach is a macroscopic one — the details of the mechanisms involved will be treated later.

2.1 Emission

Emission is described by the spontaneous *emission coefficient* j_ν. This quantity is referred to as the *volume emissivity* or just *emissivity* in some books. The emission coefficient is the energy emitted per unit volume, per unit time, per unit solid angle and per unit frequency range:

$$dE = j_\nu \, dV \, dt \, d\Omega \, d\nu. \tag{2.1}$$

For an isotropic emitter we can also define the radiated power per unit volume, per unit frequency:

$$P_\nu = 4\pi j_\nu \tag{2.2}$$

The spontaneous coefficient does not depend on the intensity of the radiation at the place where the emission occurs. Later we will meet a *stimulated* emission coefficient which does.

An alternate description of spontaneous emission makes use of the *specific emissivity* (or again just *emissivity*) ϵ_ν, defined for an isotropic emitter by:

$$dE = \epsilon_\nu \rho \, dV \, dt \, d\nu \frac{d\Omega}{4\pi} \tag{2.3}$$

where ρ is the density at the source, i.e. ϵ_ν is the energy per unit mass, per unit time and per unit frequency. Comparing (2.1) and (2.3), we have (for an isotropic emitter):

$$j_\nu = \frac{\epsilon_\nu \rho}{4\pi}. \tag{2.4}$$

It should be noted that there is some variety in terminology in the literature in this area.

Consider a beam which travels a distance ds in a time dt through an emitting material. If the cross section of the beam is dA, then the volume traversed is $dA\, ds$, and the energy added to the beam in the time dt is $dE = j_\nu dA \, ds \, dt \, d\Omega \, d\nu$, where $d\Omega$ is the element of solid angle describing the beam. All of this energy will leave the volume $dA\,ds$ in time dt, and so if the intensity associated with the added energy is dI_ν, we have from (1.3) that $dE = dI_\nu dA\,d\Omega\,dt\,d\nu$, and hence

$$dI_\nu = j_\nu ds. \tag{2.5}$$

Hence j_ν can be viewed as the rate of change of intensity with distance.

2.2 Absorption

The *absorption coefficient* (or *volume opacity*) is designated α_ν (sometimes k_ν), and has units m^{-1}. The definition of the absorption coefficient is that the change in intensity dI_ν of a beam with intensity I_ν in traversing a distance ds in a medium with absorption coefficient α_ν is

$$dI_\nu = -\alpha_\nu I_\nu ds. \tag{2.6}$$

Since absorption produces a loss in intensity, dI_ν is negative, and the negative sign in equation (2.6) means α_ν is a positive quantity. There are two ways of thinking of α_ν. It can be thought of as the fractional loss in intensity per unit distance or, for an individual photon, it is the probability of absorption per unit distance.

To understand (2.6), consider a macroscopic model in which there are n randomly distributed particles per unit volume, each presenting an absorption cross section σ_ν (units m^2) to incident radiation. If the inter-particle distance is large compared with σ_ν, then the total cross section in a volume $dA\,ds$ is $dA_{\mathrm{abs}} = n\,dA\,ds\,\sigma_\nu$. Thus the energy absorbed by a beam traversing the medium is, from (1.3)

$$I_\nu dA_{\mathrm{abs}} d\Omega\,dt\,d\nu = I_\nu (n\,dA\,ds\,\sigma_\nu)\,d\Omega\,dt\,d\nu. \tag{2.7}$$

The absorbed energy must also equal $-dI_\nu dA\,d\Omega\,dt\,d\nu$, and so have we have

$$dI_\nu = -n\sigma_\nu\,I_\nu\,ds. \tag{2.8}$$

Comparing (2.6) and (2.8), we have established that

$$\alpha_\nu = n\sigma_\nu. \tag{2.9}$$

The quantity σ_ν is also called the opacity per particle.

An alternative notation involves the *mass absorption coefficient* (or *mass opacity)* κ_ν, which has units of $\mathrm{m}^2\,\mathrm{kg}^{-1}$. This is related to the absorption coefficient by: $\alpha_\nu = \rho\kappa_\nu$.[1]

A comment in passing. The quantity $1/\alpha_\nu$ has dimensions of length and you may ask yourself: does this length have any physical significance? The answer is yes it does, it corresponds to the *mean free path* of the photons. This is the average distance a photon will travel between successive collisions. We will derive this relation shortly.

2.3 The radiative transfer equation

The radiative transfer equation gives the change in I_ν with distance. If only absorption and emission are involved, then equations (2.5) and (2.6) lead to:

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu. \tag{2.10}$$

Note that the first term depends on I_ν, but the second does not.

[1] In consulting text books on emission or absorption you will need to check from the context as to whether mass or volume units are being used. It may not always be stated explicitly.

The general problem then becomes that of solving this equation. In practice it may be necessary to resort to numerical methods. For the moment we consider two cases with simple solutions.

1. **No absorption:** when $\alpha_\nu = 0$ we have

$$\frac{dI_\nu}{ds} = j_\nu, \tag{2.11}$$

which is integrable and has the solution:

$$I_\nu = I_\nu(s_0) + \int_{s_0}^{s} j_\nu(s')\, ds'. \tag{2.12}$$

Hence the increase in brightness is equal to the emission coefficient integrated along the ray.

2. **No emission:** when $j_\nu = 0$ we have

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu, \tag{2.13}$$

which is also directly integrable, with the solution:

$$I_\nu = I_\nu(s_0) \exp\left[-\int_{s_0}^{s} \alpha_\nu(s')\, ds'\right]. \tag{2.14}$$

The brightness decreases exponentially with distance along the ray.

2.4 Optical depth

The transfer equation takes a simpler form if we introduce the *optical depth* τ_ν by the relation $d\tau_\nu = \alpha_\nu\, ds$. In integral form the definition is

$$\tau_\nu(s) = \int_{s_0}^{s} \alpha_\nu(s')ds'. \tag{2.15}$$

Despite its name, optical depth is a dimensionless quantity. Also note that in some literature τ_ν is defined to be a decreasing function of s, i.e. a definition is given that differs from (2.15) by a minus sign.

A medium is said to be *optically thick* or *opaque* when $\tau_\nu > 1$ (the integration in the definition occurs along a typical path in the medium). A medium is *optically thin* or *transparent* if $\tau_\nu < 1$. Equation (2.14) indicates that when $\tau_\nu = 1$, incident radiation would be attenuated by a factor of e^{-1} (in the absence of further emission). For an individual photon, the probability is e^{-1} of not being absorbed. This may help to give you a mental picture of what optical depth is. Alternatively, recall that the absorption coefficient is the reciprocal of the photon mean free path. Using this idea in equation (2.15) indicates that the optical depth is the ratio of the distance travelled to the photon mean free path.

The transfer equation [(2.10)] can now be written

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu, \tag{2.16}$$

where S_ν is the *source function*, defined as the ration of the emission to the absorption coefficients:

$$S_\nu \equiv \frac{j_\nu}{\alpha_\nu}. \tag{2.17}$$

In this formulation I_ν and S_ν are both functions of τ_ν. Assuming that S_ν is a known function of τ_ν, Equation (2.17) is a linear first order ODE, and can be solved by multiplying both sides by the integrating factor $\exp(\tau_\nu)$ and integrating. The formal solution of the transfer equation is:

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + \int_0^{\tau_\nu} e^{-(\tau_\nu - \tau'_\nu)} S_\nu(\tau'_\nu)\, d\tau'_\nu \tag{2.18}$$

The final intensity is seen to be the initial intensity diminished by absorption, plus the source function along the path diminished by absorption.

In the case that S_ν is constant (remember this means a constant ratio of emission to absorption), Equation (2.18) becomes

$$\begin{aligned} I_\nu(\tau_\nu) &= I_\nu(0)e^{-\tau_\nu} + S_\nu(1 - e^{-\tau_\nu}) \\ &= S_\nu + e^{-\tau_\nu}\left[I_\nu(0) - S_\nu\right]. \end{aligned} \tag{2.19}$$

This tells us something interesting. As $\tau_\nu \to \infty$, (as the radiation travels further), $I_\nu \to S_\nu$. This has a simple physical interpretation: given a sufficiently large optical depth the initial intensity becomes irrelevant, and the final intensity is determined only by the source function of the intervening material. More generally, from Equation (2.16) we see that if $I_\nu < S_\nu$ then I_ν will increase along the ray. If $I_\nu > S_\nu$ then I_ν will decrease along the ray. The source function is the quantity that the specific intensity tends to approach, and will equal if given sufficient optical depth.

2.5 Kirchoff's law

Consider an element of thermally emitting material at a temperature T placed inside an enclosure that is itself a blackbody at temperature T. Let the source function of the material be S_ν. The radiation field in which the material initially sits has specific intensity $I_\nu = B_\nu(T)$, the Planck spectrum. From the arguments in the previous section we know that if $S_\nu > B_\nu$, then I_ν will increase along a ray, and depart from the Planck spectrum. If $S_\nu < B_\nu$ then I_ν will decrease along a ray, and again depart from the Planck spectrum. However, the enclosure plus the material is also a blackbody enclosure, and so the radiation field cannot depart from $B_\nu(T)$. Hence we conclude that

$$S_\nu = B_\nu(T), \tag{2.20}$$

or

$$j_\nu = \alpha_\nu B_\nu(T). \tag{2.21}$$

Relation (2.21) is known as Kirchoff's law, and relates the emission and absorption of a thermal emitter.

The transfer equation for thermal radiation is then

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + \alpha_\nu B_\nu(T), \tag{2.22}$$

or

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + B_\nu(T). \tag{2.23}$$

For a blackbody enclosure $I_\nu = B_\nu(T)$ throughout, which trivially satisfies (2.23). It is important to draw a distinction between blackbody radiation, i.e. a radiation field that is the Planck spectrum, $I_\nu = B_\nu$, and thermal radiation, i.e. a radiation field where the source is the Planck spectrum, $S_\nu = B_\nu$. According to (2.19), thermal radiation becomes blackbody radiation for optically thick media.

Finally, note that for a thermal emitter where the temperature is a specified function of position and hence optical depth, the source function $S_\nu = B_\nu$ is a function of optical depth and hence the formal solution to the transfer equation (2.18) may be applied.

2.6 Mean free path

The *mean free path* (symbol l_ν) is the average distance a photon travels before absorption. Combining Equations (2.14) and (2.15) we have

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu}. \tag{2.24}$$

If we recognise that the intensity is proportional to the number of photons, Equation (2.24) may be interpreted as saying that the probability of a photon travelling an optical depth τ_ν is $e^{-\tau_\nu}$. The average optical depth traversed by a photon is obtained by integrating over all possible optical depths multiplied by their probability:

$$\langle\tau_\nu\rangle = \int_0^\infty \tau_\nu e^{-\tau_\nu} d\tau_\nu = 1. \tag{2.25}$$

In other words the average physical distance travelled by a photon corresponds to an optical depth of unity. For a medium where α_ν is constant, Equation (2.15) becomes $\tau_\nu = \alpha_\nu s$, where s is the physical distance travelled through the medium. Equation (2.25) then becomes

$$\langle\tau_\nu\rangle = \alpha_\nu\langle s\rangle = 1, \tag{2.26}$$

and identifying $\langle s\rangle$ with the mean free path l_ν, we have

$$l_\nu = \alpha_\nu^{-1}, \tag{2.27}$$

i.e. the mean free path is the reciprocal of the absorption coefficient, as stated at the end of Section 2.2. Finally, replacing α_ν by $1/l_\nu$ in our definition of τ_ν establishes formally that the optical depth is the ratio of the distance travelled to the mean free path:

$$\tau_\nu = \frac{s}{l_\nu}. \tag{2.28}$$

If the absorption coefficient is not constant, then the above analysis is not strictly correct. However we can still speak of a local mean free path.

The paths of photons being randomly absorbed and re-emitted in a medium provides an example of *diffusion*. Diffusion is frequently treated by analysing a statistical problem known as the *random walk*. Basically this asks if you take a step forward or backward at random, where will you be after N steps? The answer is that your average position after N steps will be your starting position, since the problem is symmetrical. However, your absolute distance from the starting point tends to increase with N. In fact, the average value of D_N^2 (the square of the distance from the starting point after N steps) is Nl^2, where l is the length of one step. To see this, consider the situation after N steps. We have $D_N = D_{N-1} \pm l$, where the two possibilities occur with equal probability. Hence we have

$$D_N^2 = \begin{cases} D_{N-1}^2 + 2lD_{N-1} + l^2 \\ \text{or} \\ D_{N-1}^2 - 2lD_{N-1} + l^2, \end{cases} \tag{2.29}$$

where the two possibilities on the right of (2.29) occur with equal probability. We require the average value of D_{N-1}^2. This is obtained by adding the averages of the expressions on the right of (2.29), with a factor of one half to represent the equal likelihood of the two outcomes. Using the fact that $\langle lD_{N-1}\rangle = l\langle D_{N-1}\rangle = 0$ (by symmetry), this gives

$$\langle D_N^2\rangle = \langle D_{N-1}^2\rangle + l^2. \tag{2.30}$$

Next note that $\langle D_1^2\rangle = l^2$. Repeated application of (2.29) then gives that $\langle D_2\rangle = 2l^2$, $\langle D_3\rangle = 3l^2$, etc., i.e.

$$\langle D_N^2\rangle = Nl^2, \tag{2.31}$$

as required. Where an average distance from the origin is required it is cutomary to use the root-mean-square distance $D_{\rm rms} = \langle D_N^2 \rangle^{1/2}$, and the preceding discussion establishes that

$$D_{\rm rms} = l\sqrt{N}. \tag{2.32}$$

Using these results we can estimate the number of collisions a photon is likely to make in moving through matter. If the distance is L and the mean free path is l, then the probable number of photon collisions is

$$N = \frac{L^2}{l_\nu^2} = \tau_\nu^2, \tag{2.33}$$

using (2.28).

As an example of an application of these ideas, consider the interior of the Sun. The solar interior is optically thick: in the solar core the mean free path for photons is $l \sim 10^{-2}$ m, whereas the distance through the medium is the solar radius $R_\odot = 6.96 \times 10^8$ m. Hence in reaching the photosphere (above which level the atmosphere is optically thin, at least in the visible), a typical photon has undergone $N = (R_\odot/l)^2 \sim 10^{21}$ absorptions and reemissions! In the core of the Sun, the average energy per photon is of order a few keV, and so the photons are X-rays. By the time the photons reach the surface they have been degraded to the visible, because as we have seen the output of the Sun peaks in the yellow part of the visible spectrum.

Equation (2.33) gives an order of magnitude estimate if the medium is optically thick. For the optically thin case the mean free path is larger than the distance through the medium, and the random walk analysis is not appropriate. In this case we can make an estimate as follows. Based on equation 2.24 we can say that the probability of at least one collision in travelling a distance with optical depth $\tau_\nu \ll 1$ is

$$\frac{I_\nu(0) - I_\nu(\tau_\nu)}{I_\nu(0)} = 1 - e^{-\tau_\nu} \approx \tau_\nu, \tag{2.34}$$

where the series expansion of the exponential has been used, and only the lowest order terms have been retained. Since τ_ν is assumed to be small, the possibility of two collisions in the given optical depth can be neglected, and then (2.34) represents the probability of one collision. It then follows that the average number of collisions is

$$N = 0 \times (1 - \tau_\nu) + 1 \times \tau_\nu = \tau_\nu. \tag{2.35}$$

Comparing Equations (2.33) and (2.35), we see that the dependence of the average number of collisions on the optical depth is different in the two regimes. To reiterate:

$$N = \begin{cases} \tau_\nu & \text{if } \tau_\nu \ll 1 \\ \tau_\nu^2 & \text{if } \tau_\nu \gg 1. \end{cases} \tag{2.36}$$

2.7 Scattering

So far we have not considered the third process for changing the energy associated with a ray: scattering. Scattering produces emission from an element of material in a radiation field. As distinct from thermal emission, scattering depends entirely on the amount of radiation falling on the element. We will consider isotropic scattering (the scattered radiation is emitted equally into all solid angles). We identify the emission coefficient associated with scattering as being proportional to the average intensity of the radiation field:

$$j_\nu = \sigma_\nu J_\nu. \tag{2.37}$$

The factor σ_ν is the *scattering coefficient*. We can recognise (2.37) as a statement of conservation of energy: the left hand side describes the power emitted per unit volume, per unit frequency range and per unit solid angle by the element, and the right hand side represents the power taken out of the radiation field in the element per unit volume, per unit frequency range and per unit solid angle. With this interpretation the right hand side represents absorption, and we can identify σ_ν as the absorption coefficient of the scattering process. Hence we can write down the source function for scattering:

$$S_\nu = \frac{j_\nu}{\sigma_\nu} = J_\nu. \tag{2.38}$$

The transfer equation for pure scattering is then

$$\begin{aligned} \frac{dI_\nu}{ds} &= -\sigma_\nu \left(I_\nu - J_\nu\right) \\ &= -\sigma_\nu I_\nu + \frac{\sigma_\nu}{4\pi}\int I_\nu d\Omega. \end{aligned} \tag{2.39}$$

Equation (2.39) is an integro-differential equation, and presents a difficult mathematical problem. We see that scattering makes the problem much more difficult, because the source function depends on the radiation field and is not specified a priori, as was assumed in Equation (2.18).

When there is both thermal emission and scattering in a medium, it is necessary to introduce an absorption coefficient describing the thermal emission as well as the scattering coefficient σ_ν. The transfer equation becomes

$$\begin{aligned} \frac{dI_\nu}{ds} &= -\alpha_\nu \left(I_\nu - B_\nu\right) - \sigma_\nu \left(I_\nu - J_\nu\right) \\ &= -\alpha_\nu^{\text{eff}} \left(I_\nu - S_\nu\right), \end{aligned} \tag{2.40}$$

where we have introduced the source function

$$S_\nu = \frac{\alpha_\nu B_\nu + \sigma_\nu J_\nu}{\alpha_\nu + \sigma_\nu}, \tag{2.41}$$

and the effective or net absorption coefficient

$$\alpha_\nu^{\text{eff}} = \alpha_\nu + \sigma_\nu. \tag{2.42}$$

The new source function is an average of the individual source functions, weighted by their respective absorption coefficients. Equation (2.40) is again an integro-differential equation, and hence is difficult to solve.

2.8 The Einstein coefficients

Kirchoff's law indicates a relation between a body's emission and absorption properties, suggesting some relationship between the processes at an atomic level. Einstein's analysis of the interaction between radiation and an atom goes as follows. Consider a simple two level atom in which level one has energy E_1 and level two has energy $E_2 = E_1 + h\nu_0$. A photon is absorbed in going from the lower to higher level and is emitted in going from the higher to the lower.

Einstein identified three possible processes:

1. Spontaneous emission: Described by the coefficient A_{21}, which is the probability per unit time that an atom in level two will drop to level one.

2. Absorption: This is a little more complicated, since absorption depends on both the atom and the incident radiation. If J_ν is the mean intensity associated with a frequency ν, then the probability of absorption per unit time is $J_\nu B_{12}$.

3. Stimulated Emission: Einstein found that in order to obtain the Planck law, an additional type of emission was required, which depended on the photon density. This is described in a similar manner to absorption, i.e. $J_\nu B_{21}$ is the probability per unit time of a transition from level two to level one due to stimulated emission. Stimulated emission has an intriguing property. A photon emitted by stimulated emission is coherent with the stimulating photon. i.e. it travels in the same direction and has the same phase.

The Einstein coefficients A_{21}, B_{12} and B_{21} give an alternative description of emission and absorption, and it is possible to derive relationships between the various coefficients, as follows.

Suppose that the number of atoms per unit volume in states one and two is n_1 and n_2 respectively. In equilibrium, these numbers will be constant. Thus the number of transitions from level one to level two per unit volume per unit time must equal the total rate of transitions per unit volume in the other direction:

$$n_1 B_{12} J_\nu = n_2 A_{21} + n_2 B_{21} J_\nu. \tag{2.43}$$

We also know from statistical mechanics that the numbers of atoms in the two states is related by:

$$\frac{n_2}{n_1} = \frac{\exp(-E_2/k_B T)}{\exp(-E_1/k_B T)} = \exp(-h\nu_0/k_B T) \tag{2.44}$$

Substituting equation (2.44) into (2.43) leads to

$$J_\nu = \frac{A_{21}/B_{21}}{(B_{12}/B_{21})\exp(h\nu_0/k_B T) - 1} \tag{2.45}$$

Equation (2.45) must equal the Planck function, and by comparison with (1.23) we have the Einstein relations

$$B_{12} = B_{21} \tag{2.46}$$

and

$$A_{21} = \frac{2h\nu^3}{c^2} B_{21}. \tag{2.47}$$

How are the Einstein coefficients related to the macroscopic absorption and emission coefficients? Recall that the amount of energy emitted in volume dV, solid angle $d\Omega$, frequency range $d\nu$ and time dt is $dE = j_\nu dV\, d\Omega\, d\nu\, dt$. If each atom contributes an energy $h\nu_0$ spread over 4π steradians for each transition, then this must also equal

$$dE = \frac{h\nu}{4\pi}\delta(\nu - \nu_0) n_2 A_{21}\, dV\, d\Omega\, d\nu\, dt, \tag{2.48}$$

where the delta function $\delta(\nu - \nu_0)$ imposes the requirement that the transition occurs only at the frequency ν_0. Equating the two expressions gives

$$j_\nu = \frac{h\nu}{4\pi}\delta(\nu - \nu_0) n_2 A_{21}. \tag{2.49}$$

To obtain the absorption coefficient, note that the energy absorbed in the frequency range $d\nu$, solid angle $d\Omega$, time dt and volume dV will be

$$dE = \frac{h\nu}{4\pi}\delta(\nu - \nu_0) n_1 B_{12} I_\nu dV\, d\Omega\, d\nu\, dt, \tag{2.50}$$

where I_ν appears instead of J_ν because we are considering absorption associated with a small solid angle $d\Omega$. Taking the volume element to be $dA\,ds$ and comparing (2.50) with (1.3) and (2.6) gives

$$\alpha_\nu = \frac{h\nu}{4\pi}\delta(\nu - \nu_0) n_1 B_{12}. \tag{2.51}$$

We have neglected stimulated emission. The name suggests that this effect should be included as a correction to the emission coefficient (2.49), but because it is proportional to the intensity, stimulated emission is best treated as negative absorption. Following an argument analogous to that leading to (2.51) gives the absorption coefficient including stimulated emission as

$$\alpha_\nu = \frac{h\nu}{4\pi}\delta(\nu - \nu_0)\left(n_1 B_{12} - n_2 B_{21}\right). \tag{2.52}$$

This equation together with (2.46) indicates that if $n_2 > n_1$ (if there is a *population inversion*), then the absorption coeffcient is negative, meaning that I_ν will increase along a ray. This is the basis of laser and maser operation. Maser emission is observed in gas clouds, from supernova remnants and from stars.

2.9 Limb darkening

There are many applications of radiative transport in astrophysics. To conclude this chapter an example is given, namely an explanation of why the limb of the Sun appears darker than the centre of the disk, an effect known as *limb darkening*.

We begin by putting the formal solution of the radiative transport equation into a different form. Rearranging (2.18) and taking the limit $\tau_\nu \to -\infty$ gives

$$I_\nu(0) = \int_{-\infty}^{0} e^{\tau'_\nu} S(\tau'_\nu)\, d\tau'_\nu. \tag{2.53}$$

We will interpret this equation as describing the intensity $I_\nu(0)$ emerging from the Sun's surface due to the source function $S_\nu(\tau_\nu)$ of material beneath the surface. Because of our choice of the surface as the point where $\tau_\nu = 0$, the integral runs over negative values of the optical depth.

The temperature of material inside the Sun increases radially with distance into the Sun. To a first approximation we can assume a simple linear increase with radial optical depth,

$$S(\tau_\nu) = a - b\,\tau_\nu \cos\theta, \tag{2.54}$$

where θ is the angle subtended by a ray at the surface to the centre of the Sun, as shown in Figure 2.1.

Introducing the radial optical depth $t_\nu = \tau_\nu \mu$ where $\mu = \cos\theta$ and substituting (2.54) into (2.53) leads to an expression for the intensity $I_\nu(0, \mu)$ of emerging radiation corresponding to an angle cosine μ:

$$I_\nu(0, \mu) = \frac{1}{\mu}\int_{-\infty}^{0} e^{t_\nu/\mu}\left(a - b\,t_\nu\right)\, dt_\nu. \tag{2.55}$$

Evaluating the integral gives

$$I_\nu(0, \mu) = a + b\mu. \tag{2.56}$$

Hence we see that $I_\nu = a + b$ at the centre of the disk, and $I_\nu = a$ at the limb: the intensity is greatest at disk centre. This result has a simple physical interpretation: most of the observed radiation along a line of sight comes from the region $-1 < \tau_\nu < 0$.

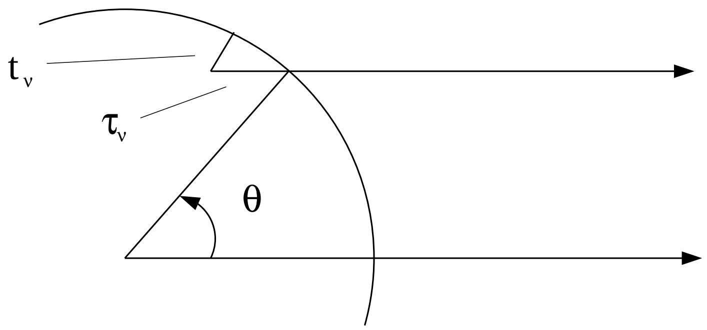


Figure 2.1: Geometry to explain limb darkening.

Because of the geometric effect illustrated by Figure 2.1, $\tau_\nu = -1$ correponds to a deeper value of t_ν, and hence to material at a hotter temperature, when the radiation comes from close to disk centre. Hence you are seeing more of the hotter underlying material when observing close to disk centre.

Although this model is crude, it provides a reasonable approximation to the limb darkening observed on the Sun.

Problem Set 2

1. Consider a thermally emitting sphere with a radius R, temperature T and absorption coefficent α_ν.

   (a) If the source is optically thin, calculate the total power produced per unit frequency by emitters in the sphere and divide by the surface area to show that the flux at the surface of the sphere is

   $$(F_\nu)_{\rm thin} = \frac{4\pi}{3}\Delta\tau_\nu B_\nu(T),$$

   where $\Delta\tau_\nu = \alpha_\nu R$ is a characteristic optical depth for the sphere.

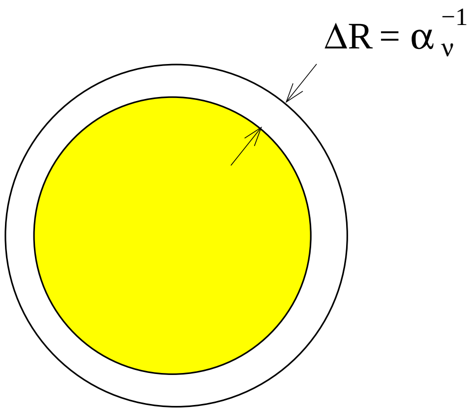


Figure 2.2: Understanding emission from an optically thick sphere.

   (b) Next assume the source is optically thick. We receive emission mostly from a shell of thickness ΔR, where $\alpha_\nu \Delta R = 1$, as shown in Figure 2.2. Since the source is optically thick, $\alpha_\nu R \gg 1$, and hence $\Delta R \ll R$. Show that the total power produced per unit frequency by emitters in the shell is approximately

   $$P_{\rm shell} \approx 16\pi^2 R^2 B_\nu(T).$$

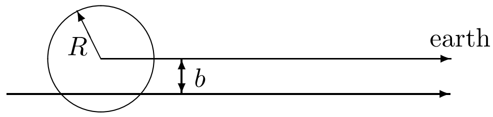
   Not all of this power escapes: if a photon heads into the region with radius less than $R - \Delta R$ then it can be assumed to be absorbed. Roughly half the photons head in, and half head out. Hence show that the flux at the surface is expected to be about

   $$(F_\nu)_{\rm thick} \approx 2\pi B_\nu(T).$$

   (c) What is the exact answer expected in part (b)? Why might the estimate in (b) be too high?

   (d) Show that the effective temperature of an optically thin source is less than that of an optically thick source.

2. A spherical cloud of gas has radius R, temperature T and is a distance d from Earth ($R << d$). The gas emits thermally at a rate P_ν (power per unit volume and per unit frequency range). Answer parts (a) to (d) for both an optically thin and an optically thick cloud.



(a) What is the brightness of the cloud as measured on Earth? Give the answer as a function of distance b from the cloud centre, assuming that the cloud can be viewed along rays parallel to a line to its centre (see diagram).

(b) What is the effective temperature of the cloud?

(c) What is the flux F_ν, as measured on earth, due to the whole cloud ?

(d) How does the cloud's measured brightness temperatures compare with the cloud's temperature?

3. Show that in the Rayleigh-Jeans limit, the transfer equation for thermal radiation can be rewritten in terms of temperature as:

$$\frac{dT_b}{d\tau_\nu} = -T_b + T.$$

Solve this equation for the case of a constant temperature T. Use the solution to determine the relationship between T_b and T for radiation emerging from a slab of emitting material that is optically thick, and for radiation emerging from a slab that is optically thin.

4. A supernova remnant has an angular diameter $\theta = 4.3$ arc minutes and a flux at 100 MHz of $F_{100} = 1.6 \times 10^{-22}$ Jm^{-2}s^{-1}Hz^{-1}. Assume that the emission is thermal.

(a) Which, if any, approximation to the Planck law might be valid at this frequency? What is the brightness temperature T_b? Was the approximation valid after all?

(b) The emitting region is actually more compact than indicated by the observed angular diameter. What effect does this have on the value of T_b?

(c) At what frequency will this object's radiation be a maximum if the emission is blackbody?

(d) What can you say about the temperature of the material from the above results?

Chapter 3

Electromagnetic Radiation I

References: Griffiths, Jackson

3.1 Review of electromagnetism

3.1.1 The Maxwell equations

Classical electromagnetism follows from the Maxwell equations, which describe the electric field $\mathbf{E} = \mathbf{E}(\mathbf{r}, t)$ and magnetic field $\mathbf{B} = \mathbf{B}(\mathbf{r}, t)$ at a point $\mathbf{r}$ in space and at a time t produced by a charge density $\rho = \rho(\mathbf{r}, t)$ and a current density $\mathbf{J} = \mathbf{J}(\mathbf{r}, t)$:

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0} & \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} & \nabla \times \mathbf{B} &= \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \end{aligned} \tag{3.1}$$

where $c^2 = (\mu_0 \epsilon_0)^{-1}$. The sources of the field, ρ and $\mathbf{J}$ are not independant, but rather are related by conservation of charge,

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}. \tag{3.2}$$

An important point to keep in mind is that these differential equations apply at a point in space — there are corresponding integral forms of the Maxwell equations which apply over extended regions of space.

3.1.2 Units

In this course we use SI units. However, many of the texts in astrophysics, to which you may need to refer, will use cgs units. The reader interested in the rationale behind the different systems of units is referred to the Appendix of Jackson's *Classical Electrodynamics*. The basic units in the cgs system are the centimetre, gram and second. The unit of force is the dyne and the unit of energy is the erg, and the conversion to Newtons and Joules has already been mentioned. More importantly in the present context, the cgs system adopts a different choice of unit for charge, the esu. The resulting system of units is also known as *Gaussian*. In Gaussian units the electromagnetic formulae take different forms to SI. For example, Coulomb's law is written as

$$F = \frac{q_1 q_2}{r^2},$$

which lacks a factor of $(4\pi\epsilon_0)^{-1}$ by comparison with the SI version. In the cgs/Gaussian system the Maxwell equations become:

$$\nabla \cdot \mathbf{E} = 4\pi\rho \qquad \nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}.$$

To convert any SI formula to Gaussian, the following prescription may be used. Make no change to q, ρ, $\mathbf{E}$, V and $\mathbf{J}$, and make the following substitutions:

$$\mathbf{B} \to \mathbf{B}/c, \quad \mathbf{A} \to \mathbf{A}/c, \quad \epsilon_0 \to \frac{1}{4\pi}, \quad \mu_0 \to \frac{4\pi}{c^2}.$$

3.1.3 Potentials

The Maxwell equations comprise a set of coupled first order partial differential equations for the components of the magnetic and electric fields. Whilst they can be solved directly in certain simple situations, it is often convenient to introduce potentials, which obey a smaller number of second order equations, and satisfy some of the Maxwell equations automatically.

A vector quantity with a vanishing divergence can be written as the curl of a vector potential, so from $\nabla \cdot \mathbf{B} = 0$ it follows that we can introduce the vector potential $\mathbf{A}$ defined by

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{3.3}$$

Substituting (3.3) into $\nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t$ gives

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = 0. \tag{3.4}$$

A vector quantity with a vanishing curl can be written as the gradient of a scalar potential, and so it follows that we can introduce the potential V defined by

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}. \tag{3.5}$$

The important point to note is that the potentials $\mathbf{A}$ and V automatically satisfy two of the Maxwell equations.

Substituting (3.3) and (3.5) into the two remaining Maxwell equations gives (exercise):

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\rho/\epsilon_0 \tag{3.6}$$

and

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla\left(\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial V}{\partial t}\right) = -\mu_0 \mathbf{J}. \tag{3.7}$$

For given electric and magnetic fields, the potentials are not uniquely defined. The only requirement on the potentials is that their derivatives reproduce the fields via equations (3.3) and (3.5). More precisely, from (3.3) it is clear that $\mathbf{B}$ will be unchanged if an arbitrary gradient of a scalar function is added to $\mathbf{A}$:

$$\mathbf{A} \to \mathbf{A}' = \mathbf{A} + \nabla\Lambda. \tag{3.8}$$

For the electric field to be unchanged under this transformation, Equation (3.5) implies that V must be modified according to

$$V \to V' = V - \frac{\partial \Lambda}{\partial t}. \tag{3.9}$$

We can exploit this arbitrariness by choosing the potentials in such a way that the equations (3.6) and (3.7) relating the potentials to the charges and currents take simple forms. One option (and the one we will be interested in here) is to choose the potentials such that the term in brackets in (3.7) vanishes:

$$\nabla \cdot \mathbf{A} = -\frac{1}{c^2}\frac{\partial V}{\partial t}, \tag{3.10}$$

a requirement known as the Lorentz condition. When this condition is met Equations (3.6) and (3.7) become a pair of symmetrical *inhomogeneous wave equations*:

$$\nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -\frac{1}{\epsilon_0}\rho, \tag{3.11a}$$

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J}. \tag{3.11b}$$

The quantities on the right in equation (3.11) involve the charges and currents which generate the potentials. They are known as the *source terms*. When the source terms are zero the equations are called *homogeneous wave equations*.

For the case of static fields ($\partial/\partial t = 0$), Equations (3.11) become Poisson's equation, and have the solutions:

$$V(\mathbf{r}_o) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}_s)}{r}\, d\tau, \tag{3.12a}$$

$$\mathbf{A}(\mathbf{r}_o) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}_s)}{r}\, d\tau. \tag{3.12b}$$

The notation in these solutions is explained in Figure 3.1. A source containing charges and currents generates potentials which are measured by some observer. There are three vectors involved. The vector $\mathbf{r}_o$ is the position vector of the observer while $\mathbf{r}_s$ is the position vector of the source. The vector $\mathbf{r}$ points from the source to the observer, and hence $r = |\mathbf{r}_o - \mathbf{r}_s|$. The integrals in equations (3.12) are volume integrals over points in the source, and the notation $d\tau$ has been chosen for a differential volume element in the source.

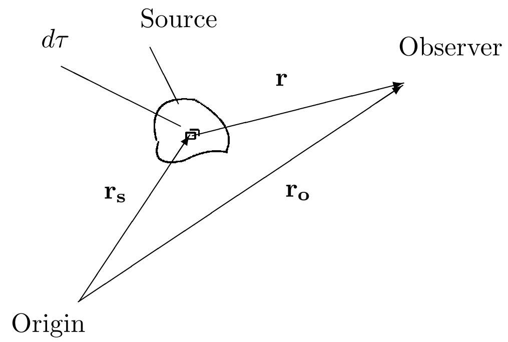


Figure 3.1: Illustration of source and observer coordinates for the solution to the Poisson equation.

3.2 Retarded potentials

What happens when the charges and currents vary with time? Equations (3.11) are wave equations with a wave speed c, and demonstrate that electromagnetic influences propagate at the speed of light. If the charges and currents vary very slowly (with respect to c), then Equations (3.12) provide a reasonable approximate solution. If not, then a solution to Equations (3.11) — the d'Alembert equation — is required.

It is possible to guess the solution. Because electromagnetic influences propagate at the speed c, the potential at some given position and time should reflect the behaviour of remote charges and currents at an earlier time. Looking at Figure 3.1, the time for the sources should be adjusted to become $t_r = t - r/c$. This time is called the *retarded time*, and with this we identify the time-dependent solutions:

$$V(\mathbf{r}_o, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}_s, t_r)}{r}\, d\tau \tag{3.13}$$

and

$$\mathbf{A}(\mathbf{r}_o, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}_s, t_r)}{r}\, d\tau. \tag{3.14}$$

3.3 Dipole radiation

As an illustration we will apply (3.13) and (3.14) to derive the potentials and hence fields for an oscillating electric dipole, and show that the dipole radiates. The exact form of the dipole is somewhat artificial, to make the derivation easier, but the final results are independent of the details. We consider two small charged spheres connected by a wire of length s. The wire is taken to lie along the z axis with its centre at the origin. The system is electrically neutral, so that the charges on the spheres at time t are $q(t)$ and $-q(t)$. Suppose that we somehow arrange to drive the charge back and forth through the wire at a frequency ω:

$$q(t) = q_0 \cos\omega t. \tag{3.15}$$

This arrangement constitutes an oscillating electric dipole. The dipole moment $\mathbf{p}(t)$ is a vector with magnitude equal to the charge times the charge separation, and with a direction pointing from the negative to the positive charge. Hence we have

$$\mathbf{p}(t) = q(t) s\,\widehat{\mathbf{z}} = p_0 \cos\omega t\,\widehat{\mathbf{z}}, \tag{3.16}$$

where $p_0 = q_0 s$ is the maximum value of the dipole moment.

The retarded potential at an observer is, according to (3.13),

$$V(\mathbf{r}_o, t) = \frac{1}{4\pi\epsilon_0} \left[\frac{q_0 \cos\omega(t - r_+/c)}{r_+} - \frac{q_0 \cos\omega(t - r_-/c)}{r_-} \right], \tag{3.17}$$

where r_+ and r_- denote the distances from the observer to the charges, as shown in Figure 3.2. According to the cosine rule,

$$r_\pm = \left[r_o^2 \mp r_o s \cos\theta + (s/2)^2 \right]^{1/2}. \tag{3.18}$$

In classical electromagnetism a distinction is made between a physical dipole, which has a finite separation, and an ideal dipole, where the separation is zero but there is a finite dipole moment. Assuming that we are dealing with an ideal dipole a number of approximations can be made, which allow the physics in (3.17) to be seen.

To obtain the limit of an ideal dipole we consider a small separation, $s \ll r_o$. Expanding (3.18) to first order in s using the Binomial theorem gives

$$r_\pm = r_o \left(1 \mp \frac{s}{2r_o} \cos\theta \right) \tag{3.19}$$

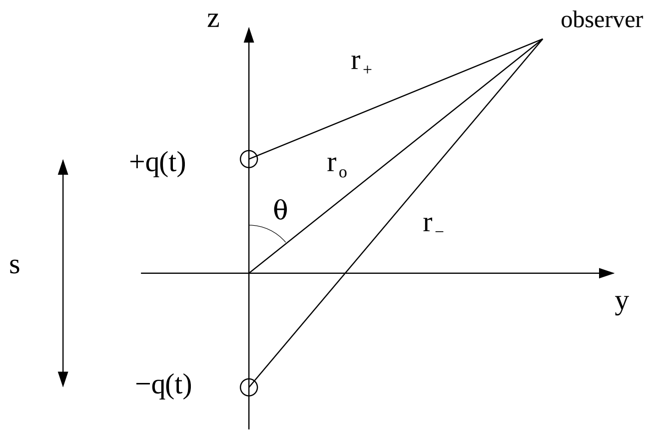


Figure 3.2: Dipole radiator.

and

$$\frac{1}{r_\pm} = \frac{1}{r_o}\left(1 \pm \frac{s}{2r_o}\cos\theta\right). \tag{3.20}$$

Using (3.19) we have

$$\begin{aligned}\cos\omega(t - r_\pm/c) &= \cos\left[\omega(t - r_o/c) \pm \frac{\omega s}{2c}\cos\theta\right] \\ &= \cos\omega(t - r_o/c)\cos\left(\frac{\omega s}{2c}\cos\theta\right) \\ &\mp \sin\omega(t - r_o/c)\sin\left(\frac{\omega s}{2c}\cos\theta\right).\end{aligned} \tag{3.21}$$

We also make the assumption that the frequency of oscillation is not very large:

$$\omega \ll \frac{c}{s}. \tag{3.22}$$

Since associated waves would have a wavelength $\lambda = 2\pi c/\omega$, this is equivalent to the assumption that $\lambda \gg s$. With this assumption we can use the small angle approximation in (3.21), to give

$$\cos\omega(t - r_\pm/c) = \cos\omega(t - r_o/c) \mp \frac{\omega s}{2c}\cos\theta\sin\omega(t - r_o/c). \tag{3.23}$$

Substituting the approximate expressions (3.20) and (3.23) into (3.17) gives

$$V(r_o,\theta,t) = \frac{p_0\cos\theta}{4\pi\epsilon_0 r_o}\left[\frac{1}{r_o}\cos\omega(t - r_o/c) - \frac{\omega}{c}\sin\omega(t - r_o/c)\right]. \tag{3.24}$$

If there is no oscillation of charge ($\omega \to 0$), then (3.24) becomes

$$V = \frac{p_0\cos\theta}{4\pi\epsilon_0 r_o^2}, \tag{3.25}$$

which is the potential for a static dipole. When there is time variation, there are two terms in (3.24) contributing to the potential. At large distances the first term varies like r_o^{-2}, whereas the second term varies like r_o^{-1}. Hence the second term falls off more slowly, and at a significant distance from the source this term will dominate. More formally, we are interested in fields which survive in the *radiation zone*, i.e. where

$$r_o \gg \frac{c}{\omega}, \tag{3.26}$$

which is equivalent to the requirement that $r_o \gg \lambda$. In the radiation zone the potential takes the simple form

$$V(r_o, \theta, t) = -\frac{p_0 \omega}{4\pi\epsilon_0 c}\left(\frac{\cos\theta}{r_o}\right)\sin\omega(t - r_o/c). \tag{3.27}$$

Next consider the vector potential $\mathbf{A}(\mathbf{r_o}, \mathbf{t})$, which is dependent on the current $I(t)$ flowing in the wire:

$$I(t) = \frac{dq}{dt} = -q_o \omega \sin\omega t. \tag{3.28}$$

From (3.14) we have

$$\mathbf{A}(\mathbf{r}_o, t) = \frac{\mu_0}{4\pi}\int_{-s/2}^{s/2} \frac{-q_o\omega\sin\omega(t - r/c)}{r}\widehat{\mathbf{z}}\, dz. \tag{3.29}$$

The integration in (3.29) effectively introduces a factor of s, and so to first order in s/r_o we can replace the integration by s times the integrand evaluated at the origin, where $r = r_o$:

$$\mathbf{A}(r_o, \theta, t) = -\frac{\mu_0 p_o \omega}{4\pi r_o}\sin\omega(t - r_o/c)\widehat{\mathbf{z}}. \tag{3.30}$$

In spherical coordinates this expression has two components, since

$$\widehat{\mathbf{z}} = \cos\theta\widehat{\mathbf{r}} - \sin\theta\widehat{\boldsymbol{\theta}}. \tag{3.31}$$

From the potentials it is straightforward to calculate the fields, using (3.3) and (3.5). Only terms important in the radiation zone (i.e. that decay like r_o^{-1}) are kept. The details are left as an exercise, but the results are:

$$\mathbf{B}(r_o, \theta, t) = -\frac{\mu_0 p_0 \omega^2}{4\pi c}\left(\frac{\sin\theta}{r_o}\right)\cos\omega(t - r_o/c)\,\widehat{\boldsymbol{\phi}} \tag{3.32}$$

and

$$\mathbf{E}(r_o, \theta, t) = -\frac{\mu_0 p_0 \omega^2}{4\pi}\left(\frac{\sin\theta}{r_o}\right)\cos\omega(t - r_o/c)\,\widehat{\boldsymbol{\theta}}. \tag{3.33}$$

Equations (3.32) and (3.33) define waves of frequency ω travelling radially outwards at the speed of light. The electric and magnetic fields are in phase, mutually perpendicular and transverse, with a ratio of amplitudes $E_0/B_0 = c$. This is the same as for plane electromagnetic waves in free space. However, the amplitudes of the dipole radiation waves decay like r_o^{-1}, because these are spherical waves, not plane waves. A wavefront instantaneously defines a sphere and not a plane.

The energy flux in the travelling waves is determined by the Poynting vector $\mathbf{S} = (\mathbf{E} \times \mathbf{B})/\mu_0$. Using (3.32) and (3.33) gives

$$\mathbf{S} = \frac{\mu_0}{c}\left[\frac{p_0\omega^2}{4\pi}\left(\frac{\sin\theta}{r_o}\right)\cos\omega(t - r_o/c)\right]^2 \widehat{\mathbf{r}}. \tag{3.34}$$

The Poynting vector gives the instantaneous flux of energy in the wavefront. To an observer measuring that flux, the relevant quantity is the time average of the flux.

Denoting the time average by angled brackets and recognising that the average of $\cos^2 x$ over many cycles is $\frac{1}{2}$, we have

$$\langle \mathbf{S} \rangle = \left(\frac{\mu_o p_o^2 \omega^4}{32\pi^2 c} \right) \frac{\sin^2 \theta}{r_o^2} \widehat{\mathbf{r}}. \tag{3.35}$$

Equation (3.35) demonstrates that there is no power radiated along the axis of the dipole ($\theta = 0$). The maximum flux occurs at $\theta = \pi/2$, i.e. at right angles to the dipole.

There is an interesting consequence of the result (3.35): the strong dependence of radiated power on frequency ($\sim \omega^4$) accounts for the blue colour of the sky. As sunlight passes through the atmosphere it causes atoms in molecules in the atmosphere to oscillate, and the molecules behave as small dipole radiators. They emit more strongly when driven at higher frequencies, and so the energy absorbed and reradiated by the atmospheric dipoles is stronger for the blue component of sunlight. The dipoles oscillate in a direction perpendicular to the Sun's rays, which also means that if you look in a direction in the sky at right angles to the Sun, the radiation you receive is polarised in a direction perpendicular to the line from the point you are looking at to the Sun. This can be confirmed with a polarising filter.

Recent colour pictures from Mars Pathfinder demonstrate that the sky on Mars is not blue but rather an orange colour, confirming a result from the 1970s Viking landers. The atmosphere of Mars is thin and dusty, and atmospheric light scattering is dominated not by molecules of gas (in the case of Mars, the atmosphere is mostly carbon dioxide) but by suspended dust particles. These particles are larger than the wavelengths of visible light, and they are reddened by iron oxide, like the Martian soil. The power spectrum of the scattered light in this case is not dictated by the dipole radiation formula but by the wavelengths which are reflected from the suspended dust.

3.4 Radiation from a moving point charge

3.4.1 Potentials due to a moving point charge

We have derived the potentials and fields produced by the movement of charge in an oscillating dipole. What about the potentials and fields due to a single charge with a specified trajectory? The potentials in this case are the famous Liénard-Wiechert potentials. The treatment given here follows Griffiths, and is not completely rigorous.

Suppose the trajectory is described by the position vector $\mathbf{w}(t)$. Figure 3.4 illustrates the path of a particle, and the position vector of the particle $\mathbf{w}(t_r)$ and the vector $\mathbf{r} = \mathbf{r}_o - \mathbf{w}(t_r)$ for the retarded time t_r corresponding to a time t. The vector $\mathbf{w}(t_r)$ is the *retarded position* of the particle. The retarded time is implicitly defined by the equation

$$r = |\mathbf{r}_o - \mathbf{w}(t_r)| = c(t - t_r). \tag{3.36}$$

A quick examination of formula (3.13), i.e.

$$V(\mathbf{r}_o, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}_s, t - r/c)}{r} \, d\tau \tag{3.37}$$

might lead you to expect that the retarded potential of a point charge is

$$V = \frac{1}{4\pi\epsilon_0} \frac{q}{r}, \tag{3.38}$$

with the understanding that the distance r is the distance from the observer to the charge at the retarded time. However, this is wrong, for a subtle reason. The denominator r

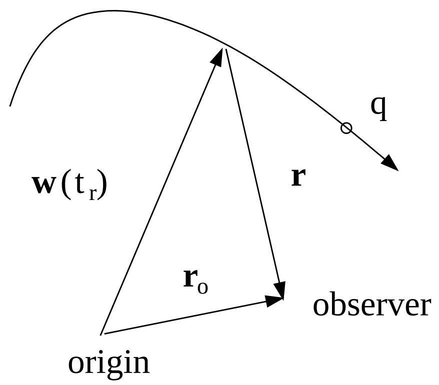


Figure 3.3: Geometry of retarded position.

can be taken outside the integral, but the remaining integral,

$$\int \rho(\mathbf{r}_s, t - |\mathbf{r}_o - \mathbf{r}_s|/c)\, d\tau \tag{3.39}$$

is not the charge of the particle. The reason is that to obtain the charge you need to integrate over the charge distribution at a fixed instant in time, but in this case the charge distribution at different source points is evaluated at different times, owing to the dependence of the retarded time on the source coordinate. It might be expected that this problem goes away for a point particle, but it does not. To see why, consider the following analogy.

If we observe a moving object, its dimensions in a direction parallel to our line of sight appear altered, because the light from different parts of the object take different times to reach us. This effect is classical — this is not relativistic contraction (in treatments of relativistic contraction this effect is actually removed by specifying that the length of an object is measured by *simultaneously* determining the positions of the endpoints of the object in the given frame of reference). Consider the motion of a rod of actual length L, as shown in Figure 3.4. We take θ to be the angle between the velocity of the rod and a unit vector $\widehat{\mathbf{r}}$ pointing from the rod to the observer (we assume the rod is far enough away that rays from the ends of the object can be assumed parallel). The observed length of the object is L'. To relate this to L, note that photons from the rear of the object have to cover an extra distance $L' \cos\theta$. Hence they must have left a time $\Delta t = L' \cos\theta/c$ earlier. In that time the rod has moved a distance $L' - L = v\Delta t$. Solving this equation for L' gives

$$L' = \frac{L}{(1 - v\cos\theta/c)} = \frac{L}{(1 - \widehat{\mathbf{r}}\cdot\mathbf{v}/c)}. \tag{3.40}$$

Since there is no motion in the direction of the thickness of the rod, the observed thickness is the actual thickness. It follows that the observed volume τ' is related to the true volume τ by the same factor:

$$\tau' = \frac{\tau}{(1 - \widehat{\mathbf{r}}\cdot\mathbf{v}/c)}. \tag{3.41}$$

Note that the correction factor does not depend on the size of the object. Hence it must also apply for a point object, i.e. a particle.

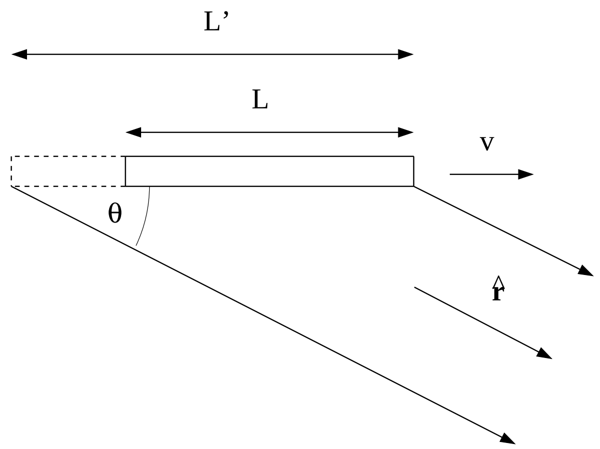


Figure 3.4: Geometry for understanding the factor in the Liénard-Wiechert potentials.

This situation is analogous to the problem at hand, namely evaluating (3.39). The effective volume of the charge is modified by the factor identified above, because for each point in the source we are putting in the charge density for an earlier time, namely the retarded time for the source point. The effect of this is to alter the apparent charge:

$$\int \rho(\mathbf{r}_s, t - |\mathbf{r}_o - \mathbf{r}_s|/c)\, d\tau = \frac{q}{(1 - \widehat{\mathbf{r}} \cdot \mathbf{v}/c)}, \tag{3.42}$$

where $\mathbf{v}$ is the velocity at the retarded time, and $\mathbf{r}$ is the vector from the retarded position to the point of observation $\mathbf{r}_o$.

The equation for the electric potential then becomes:

$$V(\mathbf{r}_o, t) = \frac{1}{4\pi\epsilon_0} \frac{q}{r\,(1 - \widehat{\mathbf{r}} \cdot \mathbf{v}/c)}. \tag{3.43}$$

Similarly the vector potential is

$$\mathbf{A}(\mathbf{r}_o, t) = \frac{\mu_0}{4\pi} \frac{q\mathbf{v}}{r\,(1 - \widehat{\mathbf{r}} \cdot \mathbf{v}/c)} = \frac{\mathbf{v}}{c^2} V(\mathbf{r}_o, t). \tag{3.44}$$

Equations (3.43) and (3.44) are the Liénard-Wiechert potentials for a moving charge.

We can use these equations to calculate the potentials from charges moving in prescribed ways, and then use equations (3.3) and (3.5) to calculate the fields.

3.4.2 Fields due to a moving point charge

The field from a moving point charge was derived in PHYS202 from relativistic arguments. It is also possible to derive these fields from the potentials (3.43) and (3.44) obtained above. We will not go through the derivation — it can be found in Griffiths, and in other books on electromagnetism – but will state the results. The derivation requires some care, because the derivatives in (3.3) and (3.5) are performed with respect to the observer's coordinates.

The results are that the fields are given by

$$\mathbf{E}(\mathbf{r}_o, t) = \frac{q}{4\pi\epsilon_0} \frac{r}{(\mathbf{r} \cdot \mathbf{u})^3} \left[(c^2 - v^2)\mathbf{u} + \mathbf{r} \times (\mathbf{u} \times \mathbf{a})\right] \tag{3.45}$$

and

$$\mathbf{B}(\mathbf{r}_o, t) = \frac{1}{c}\widehat{\mathbf{r}} \times \mathbf{E}, \tag{3.46}$$

where we have introduced the vector

$$\mathbf{u} \equiv c\widehat{\mathbf{r}} - \mathbf{v}. \tag{3.47}$$

In these equations $\mathbf{v}$ and $\mathbf{a}$ are the velocity and acceleration of the particle at the retarded time, and $\mathbf{r}$ is the vector from the retarded position of the particle to the observer.

The first term in equation (3.45) falls off as r^{-2}. This can be seen by noting the r in the numerator and the r^3 in the denominator. If $\mathbf{v} = \mathbf{a} = 0$ only this term survives, and the field reduces to the Coulomb field

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\widehat{\mathbf{r}}. \tag{3.48}$$

For this reason this term is referred to as the generalised Coulomb field, or *velocity field* (because it does not depend on the acceleration). The second term is the *acceleration field*, which falls off as r^{-1}, and hence is dominant at large distances. This term makes up the radiation field.

3.4.3 Fields from particles with constant v

While equations (3.45) and (3.46) give the fields it is not easy, by looking at these equations, to see what these fields are like (especially in the case of $\mathbf{E}$), and it is useful to reduce the expressions in a simple case. We will consider the case of a particle moving with a constant velocity $\mathbf{v}$.

Assuming the particle is at the origin at time $t = 0$, we have the trajectory

$$\mathbf{w} = \mathbf{v}t. \tag{3.49}$$

To obtain the retarded time, we need to solve (3.36) for this choice of trajectory, i.e. we need to solve

$$|\mathbf{r}_o - \mathbf{v}t_r| = c(t - t_r). \tag{3.50}$$

Squaring both sides gives a quadratic in t_r, which can be solved using the quadratic fomula, to give:

$$t_r = \frac{c^2 t - \mathbf{r}_o \cdot \mathbf{v} \pm \left[(c^2 t - \mathbf{r}_o \cdot \mathbf{v})^2 + (c^2 - v^2)(r_o^2 - c^2 t^2)\right]^{1/2}}{c^2 - v^2}. \tag{3.51}$$

When $\mathbf{v} = 0$ this reduces to $t_r = t \pm r_o/c$, which indicates that the minus sign is the correct choice.

Now we turn to the expressions for the fields. Setting $\mathbf{a} = 0$ in (3.45) gives

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{(c^2 - v^2)}{(\mathbf{r} \cdot \mathbf{u})^3} r\mathbf{u}. \tag{3.52}$$

The general procedure is to reduce the right hand side to a form that depends only on the observer's position $\mathbf{r}_0$, the present time t, and the velocity $\mathbf{v}$. In particular this involves using (3.51) to replace the retarded time.

First consider the term $r\mathbf{u}$:

$$\begin{aligned} r\mathbf{u} &= c\mathbf{r} - r\mathbf{v} \\ &= c(\mathbf{r}_o - \mathbf{v}t_r) - c(t - t_r)\mathbf{v}, \end{aligned} \tag{3.53}$$

using the definition of $\mathbf{u}$ and (3.36). Hence

$$r\mathbf{u} = c(\mathbf{r}_o - \mathbf{v}t). \tag{3.54}$$

Next consider the term

$$\begin{aligned}
\mathbf{r}\cdot\mathbf{u} &= cr\left(1 - \frac{\widehat{\mathbf{r}}\cdot\mathbf{v}}{c}\right) \\
&= c^2(t - t_r)\left[1 - \frac{\mathbf{v}\cdot(\mathbf{r}_o - \mathbf{v}t_r)}{c^2(t - t_r)}\right] \\
&= c^2 t - \mathbf{r}_o\cdot\mathbf{v} - t_r(c^2 - v^2),
\end{aligned} \tag{3.55}$$

where we have used (3.36). Replacing the t_r term using (3.51) gives

$$\mathbf{r}\cdot\mathbf{u} = \left[(c^2 t - \mathbf{r}_o\cdot\mathbf{v})^2 + (c^2 - v^2)(r_o^2 - c^2t^2)\right]^{1/2}. \tag{3.56}$$

Introducing the vector from the *present* position of the particle to the observer,

$$\mathbf{R} \equiv \mathbf{r}_o - \mathbf{v}t, \tag{3.57}$$

it is straightforward to show that (3.56) can be rewritten

$$\mathbf{r}\cdot\mathbf{u} = cR\left[1 - \frac{v^2}{c^2}\sin^2\theta\right]^{1/2}, \tag{3.58}$$

where θ is the angle between $\mathbf{R}$ and $\mathbf{v}$. Figure 3.5 illustrates these definitions.

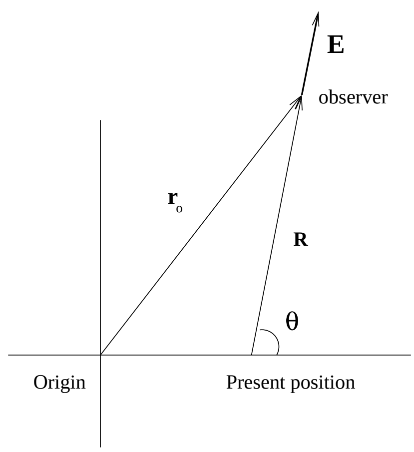


Figure 3.5: Diagram of the quantities involved in equation (3.59)

Substituting the expressions (3.54) and (3.56) into (3.52) gives

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{\widehat{\mathbf{R}}}{R^2}\frac{\left(1 - v^2/c^2\right)}{\left[1 - (v/c)^2\sin^2\theta\right]^{3/2}} \tag{3.59}$$

The magnetic field is obtained from (3.46). Noting that

$$\begin{aligned}
\widehat{\mathbf{r}} &= \frac{\mathbf{r}_0 - \mathbf{v}t_r}{r} \\
&= \frac{(\mathbf{r}_o - \mathbf{v}t) + \mathbf{v}(t - t_r)}{r} \\
&= \frac{\mathbf{R}}{r} + \frac{\mathbf{v}}{c},
\end{aligned} \tag{3.60}$$

we have that

$$\mathbf{B} = \frac{1}{c}\hat{\mathbf{r}} \times \mathbf{E} = \frac{1}{c^2}(\mathbf{v} \times \mathbf{E}). \tag{3.61}$$

Equations (3.59) and (3.61) are the expressions we are after. There is a strange aspect to these answers: Equation (3.59) indicates that **E** is in the direction of **R**, and so points away from the *present* position of the particle, even though the field is generated by the *retarded* potential. Evidently the field 'predicts' the future position of the particle! However it should be noted that if the velocity had changed during the time between t_r and t, the field (at large distance) would point away from the position where the charge actually ends up: there is nothing prescient about the field.

Equation (3.59) differs from the Coulomb formula for the field due to a stationary charge by the factors in the numerator and denominator. The angular dependence of the field is determined by the $\sin^2\theta$ term in the denominator. Clearly the field is strongest for $\theta = \pi/2$, i.e. at right angles to the motion of the particle: the field lines 'bunch up' in this direction. Figure 3.6 illustrates the electric field of the particle.

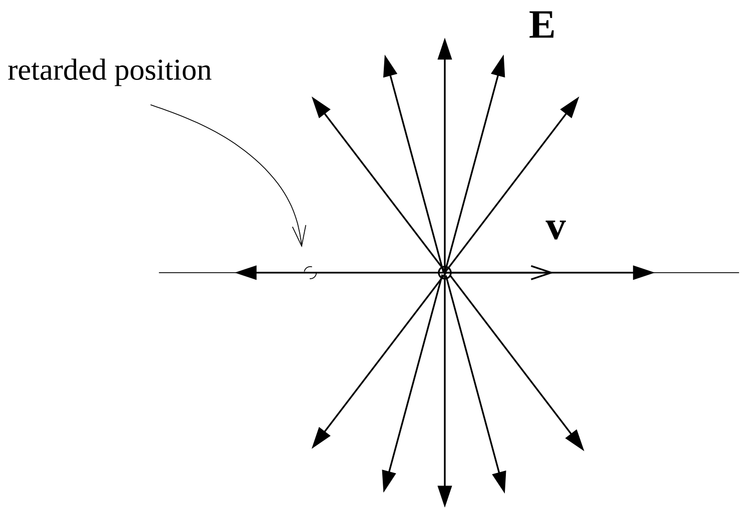


Figure 3.6: Electric field of a particle moving with a constant velocity.

Problem Set 3

1. Fill in the steps in the derivations of (3.6) and (3.7).
2. Derive the dipole radiation fields (3.32) and (3.33) from the potentials (3.27) and (3.30).
3. Show that the total power radiated by an oscillating dipole is independent of distance from the dipole.
4. Establish (3.58) from (3.56)
5. What does the magnetic field due to a charge in motion with a uniform velocity look like?
6. Suppose that a point charge moves along the x axis with a constant speed v. Show that the fields at points to the right of the charge are given by

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0 r^2}\left(\frac{c+v}{c-v}\right)\widehat{\mathbf{x}}, \quad \mathbf{B} = \mathbf{0},$$

where (as defined in the notes) $\mathbf{r}$ is the vector from the retarded position of the charge to the observation point. What are the fields to the left of the charge ?

Chapter 4

Electromagnetic Radiation II

References: Griffiths, Rybicki and Lightman

4.1 The Larmor formula

4.1.1 Power radiated by a point charge

We have an expression for the acceleration part of the electric field produced by a point charge [the second term in (3.45)]:

$$\mathbf{E}_{\rm rad} = \frac{q}{4\pi\epsilon_0}\frac{r}{(\mathbf{r}\cdot\mathbf{u})^3}\left[\mathbf{r}\times(\mathbf{u}\times\mathbf{a})\right], \tag{4.1}$$

where $\mathbf{r}$ is the vector from the retarded position of the source to the observer, $\mathbf{a}$ is the acceleration of the charge at the retarded time and $\mathbf{u} = c\hat{\mathbf{r}} - \mathbf{v}$, where $\mathbf{v}$ is the velocity at the retarded time.

The energy flux associated with the fields of a point charge is given by the Poynting vector,

$$\begin{aligned}\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) &= \frac{1}{\mu_0 c}\left[\mathbf{E}\times(\hat{\mathbf{r}}\times\mathbf{E})\right] \\ &= \frac{1}{\mu_0 c}\left[\hat{\mathbf{r}}E^2 - \mathbf{E}(\hat{\mathbf{r}}\cdot\mathbf{E})\right], \end{aligned}\tag{4.2}$$

using (3.46). We wish to determine the power radiated by the point charge. This is obtained by integrating (4.2) over a sphere centred on the retarded position of the particle, with radius r. In doing this integration there are contributions from both the acceleration and velocity fields of the point charge. However, as established in the previous chapter, the velocity field falls off like r^{-2}, and so the resulting contribution to the Poynting vector falls off like r^{-4}. Integrating over a sphere of radius r, the resulting contribution to the power decays like r^{-2}, and hence there is no power radiated to infinity from the velocity field. This is not the case with the acceleration field, Equation (4.1). This field decays like r^{-1}, the associated Poynting flux decays like r^{-2}, and the total power radiated is independent of distance. Hence the acceleration field radiates power to infinity, which is why it is called the radiation field. A charge *must* accelerate to radiate energy to infinity. As a result we restrict our attention to the acceleration field, and the Poynting vector simplifies to

$$\mathbf{S}_{\rm rad} = \frac{1}{\mu_0 c}E_{\rm rad}^2\hat{\mathbf{r}}. \tag{4.3}$$

Next we assume that the charge is at rest at the retarded time, so that $\mathbf{v} = \mathbf{0}$ (but $\mathbf{a} \neq \mathbf{0}$). (The results obtained below turn out to be a good approximation provided $v \ll c$, but a relativistic correction is needed for fast particles, as discussed in Chapter 5.) With this assumption $\mathbf{u} = c\widehat{\mathbf{r}}$, and we have

$$\begin{aligned} \mathbf{E}_{\text{rad}} &= \frac{q}{4\pi\epsilon_0 c^2}\frac{1}{r}\left[\widehat{\mathbf{r}} \times (\widehat{\mathbf{r}} \times \mathbf{a})\right] \\ &= \frac{q}{4\pi\epsilon_0 c^2}\frac{1}{r}\left[\widehat{\mathbf{r}}(\widehat{\mathbf{r}} \cdot \mathbf{a}) - \mathbf{a}\right]. \end{aligned} \tag{4.4}$$

Using (4.3) the Poynting flux is then

$$\mathbf{S}_{\text{rad}} = \frac{1}{\mu_0 c}\left(\frac{q}{4\pi\epsilon_0 c^2}\right)^2 \frac{a^2 \sin^2\theta}{r^2}\widehat{\mathbf{r}}, \tag{4.5}$$

where θ is the angle between $\widehat{\mathbf{r}}$ and $\mathbf{a}$. Equation (4.5) indicates that no radiation is emitted in the forward or backward direction, but all emission occurs in a donut about the direction of instantaneous acceleration, as shown in Figure 4.1. The angular ditribution of radiation is the same as was found for the oscillating dipole (because the dipole consists of charges accelerating back and forth in the direction of the dipole).

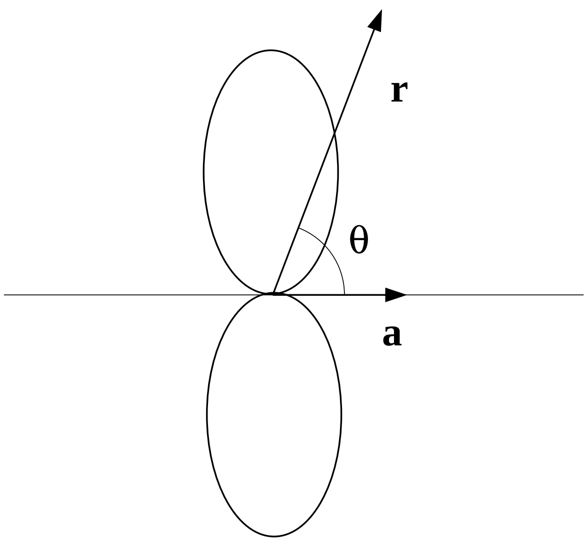


Figure 4.1: Angular distribution of radiative flux from an accelerating charge.

We can rewrite (4.5) in a way that will be useful in later discussions. By noting that the power emitted into a solid angle $d\Omega$ in the direction $\widehat{\mathbf{r}}$ is $dP = S_{\text{rad}} r^2 d\Omega$, we have the expression for the power emitted per unit solid angle:

$$\frac{dP}{d\Omega} = \frac{q^2 a^2 \sin^2\theta}{16\pi^2\epsilon_0 c^3}. \tag{4.6}$$

The total power radiated is obtained by integrating over all solid angles:

$$P = \int \frac{dP}{d\Omega} d\Omega = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{q^2 a^2}{c^3}. \tag{4.7}$$

This expression is the *Larmor formula*.

4.1.2 A graphical explanation of the acceleration field

There is an informative graphical explanation for the radiation field from a point charge, due originally to J.J. Thomson. Consider a charge that has been in motion with a constant velocity along the x axis, and then decelerates to rest at position x_0 at time t_0. The deceleration is assumed to take a short time Δt. What does the electric field produced by the particle look like at time $t_1 > t_0$? Figure 4.2 is a schematic of the field structure. Beyond a distance $r = c(t_1 - t_0 + \Delta t)$ from the charge, the field cannot know that the particle has stopped. Hence the field structure outside this radius must be that of a charge in uniform motion, viz. Figure 3.6. Note that this field has field lines radiating from the point x_1, which is where the particle would have been if it had not stopped [see discussion following Equation (3.61)]. Close to the particle the field lines must be those of the Coulomb field. In between is an annular region with thickness $c\Delta t$ corresponding to the time when the particle was decelerating. The Maxwell equation $\nabla \cdot \mathbf{E} = 0$ (for space in the absence of free charge) requires that the electric field lines of the two regions must join up, and so the field in the annular region must look something like that shown in Figure 4.2. The radial width of the annular region is fixed: the inner and outer radii propagate out at the speed of light. In between is the radiation or acceleration field. Figure 4.2 reproduces many of the important features of the acceleration field. For example, there are no kinks in the field lines along the x axis. This corresponds to the fact that there is no radiation along the direction of acceleration of the particle. The number of field lines in the annular region is fixed. The field lines inside this region are packed more closely than the radial fields outside, which corresponds to the r^{-1} decline of the radiation field. More precisely, it is apparent that the field lines depart from the radial direction in the annular region. This amounts to the addition of a component of field perpendicular to the radial direction, that we will call $B_\perp$. We can estimate $B_\perp$ as follows. Assume the annular region has (constant) width Δr, and that there are N field lines passing through the annulus in the plane shown in Figure 4.2. Then the inclination of the field lines to the radial direction in the annular region is given by $\tan\theta \approx (2\pi r/N)/\Delta r$. Noting that $B_\perp/B_0 = \tan\theta$ and that $B_0 \sim r^{-2}$, it follows that $B_\perp \sim r^{-1}$, and hence the radiation field decays like r^{-1}.

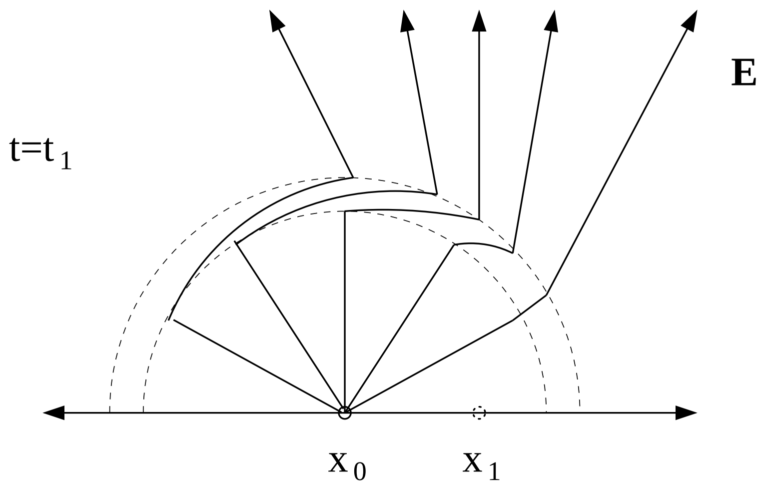


Figure 4.2: Graphical explanation of the radiation field.

4.1.3 Radiation from a group of charges - the dipole approximation

So far we have focused on a the radiation produced by an oscillating dipole, and by a single accelerating charge. What about an arbitrary collection of charges? Clearly we can always just add up (4.4) for each charge:

$$\mathbf{E}_{\rm rad} = \frac{1}{4\pi\epsilon_0 c^2}\sum_i \frac{q_i}{r_i}\left[\widehat{\mathbf{r}}_i \times (\widehat{\mathbf{r}}_i \times \mathbf{a}_i)\right], \tag{4.8}$$

where the subscripts i denote quantities associated with each charge. Provided the charges are close together (with respect to the distance to the observer), it is reasonable to assume that the distance to the charges at the retarded time can be assumed to be the same for all charges, say r_0, with an associated position vector $\widehat{\mathbf{n}}$. In that case we can write

$$\mathbf{E}_{\rm rad} = \frac{\widehat{\mathbf{n}} \times (\widehat{\mathbf{n}} \times \ddot{\mathbf{p}})}{4\pi\epsilon_0 r_0 c^2}, \tag{4.9}$$

where

$$\mathbf{p} = \sum_i q_i \mathbf{r}_i \tag{4.10}$$

is the *net dipole moment* of the collection of charges. Equation (4.9) is the *dipole approximation* (for a more careful justification of the approximation involved, see Rybicki and Lightman). Similarly (4.7) has the counterpart in the dipole approximation:

$$P = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{\ddot{p}^2}{c^3}. \tag{4.11}$$

It is possible to rewrite these formulae in a useful alternative way. First consider the magnitude of the electric field $E_{\rm rad}(t)$ due to the radiation field at a given point in space. The Fourier transform of this quantity is

$$\widetilde{E}_{\rm rad}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{\rm rad}(t) e^{i\omega t} dt. \tag{4.12}$$

Because the radiation field consists of transverse electromagnetic waves, the magnitude of the Poynting vector is $S_{\rm rad} = \epsilon_0 c E_{\rm rad}^2$ (see Problem Set 4). Recognising this as the energy per unit area and per unit time in the radiated field we can integrate over all time,

$$\frac{dW}{dA} = \epsilon_0 c \int_{\infty}^{\infty} E_{\rm rad}^2(t) dt, \tag{4.13}$$

to obtain the total energy radiated per unit area. Parseval's theorem for the Fourier transform of a real quantity:

$$\int_{-\infty}^{\infty} E_{\rm rad}^2(t) dt = 4\pi \int_0^{\infty} |\widetilde{E}_{\rm rad}(\omega)|^2 d\omega \tag{4.14}$$

allows us to rewrite (4.13) as

$$\frac{dW}{dA} = 4\pi\epsilon_0 c \int_0^{\infty} |\widetilde{E}_{\rm rad}(\omega)|^2 d\omega, \tag{4.15}$$

or

$$\frac{dW}{dA d\omega} = 4\pi\epsilon_0 c |\widetilde{E}_{\rm rad}(\omega)|^2. \tag{4.16}$$

Next we introduce the dipole approximation. From (4.9) we have

$$E_{\rm rad} = \frac{\ddot{p}\sin\theta}{4\pi\epsilon_0 c^2 r_0}, \tag{4.17}$$

where θ is the angle between $\widehat{\mathbf{n}}$ and $\ddot{\mathbf{p}}$. Taking the Fourier transform we have

$$\widetilde{E}_{\rm rad}(\omega) = \frac{-\omega^2\widetilde{p}\sin\theta}{4\pi\epsilon_0 c^2 r_0}, \tag{4.18}$$

and inserting this in (4.16) gives

$$\frac{dW}{dAd\omega} = \frac{1}{4\pi\epsilon_0}\frac{\omega^4\widetilde{p}^2\sin^2\theta}{c^3 r_0^2}. \tag{4.19}$$

Next note that the differential area into which the energy is radiated is related to solid angle by $dA = r_0^2 d\Omega$, and hence

$$\frac{dW}{d\omega d\Omega} = \frac{1}{4\pi\epsilon_0}\frac{\omega^4\widetilde{p}^2\sin^2\theta}{c^3}. \tag{4.20}$$

Integrating over solid angle leads to the total energy radiated per unit frequency,

$$\frac{dW}{d\omega} = \frac{2}{3\epsilon_0}\frac{\omega^4\widetilde{p}^2}{c^3}. \tag{4.21}$$

4.2 Thomson Scattering

As an application of the theory developed above, consider the process in which a free electron oscillates in response to the passage of a plane electromagnetic wave. We assume the incident electromagnetic wave has angular frequency ω and propagates along the x axis, with the electric field in the z direction. The electron is assume to have an equilibrium position at the origin. The electric field at the origin due to the electromagnetic wave is

$$\mathbf{E} = E_o \sin\omega t\,\widehat{\mathbf{z}}. \tag{4.22}$$

The acceleration of the electron is then $\mathbf{a} = -e\mathbf{E}/m$, and using (4.5) the Poynting flux of the radiation produced by the electron is

$$\mathbf{S}_{\rm rad} = \frac{e^4 E_0^2}{16\pi^2\epsilon_0 m^2 c^3}\frac{\sin^2\theta}{r^2}\sin^2\omega t\,\widehat{\mathbf{n}}, \tag{4.23}$$

where $\widehat{\mathbf{n}}$ is the vector describing the direction of the radiation, and $\widehat{\mathbf{n}}\cdot\widehat{\mathbf{z}} = \cos\theta$. The time averaged power radiated per unit solid angle is given by

$$\frac{dP}{d\Omega} = r^2\langle S_{\rm rad}\rangle, \tag{4.24}$$

where $\langle ...\rangle$ denotes the time average. Recalling that the time average of $\sin^2 x$ is $\frac{1}{2}$ we have

$$\frac{dP}{d\Omega} = \frac{e^4 E_0^2}{32\pi^2\epsilon_0 m^2 c^3}\sin^2\theta. \tag{4.25}$$

Next we introduce the *differential cross section* $d\sigma/d\Omega$ according to

$$\frac{dP}{d\Omega} = \mathcal{F}\frac{d\sigma}{d\Omega}, \tag{4.26}$$

where $\mathcal{F}$ is the time average flux in the electromagnetic wave incident on the electron at the origin. The differential cross section describes the probability for scattering in a given direction. Because the incident wave is a plane electromagnetic wave, we have $\mathcal{F} = \frac{1}{2}\epsilon_0 c E_0^2$ (see Problem set 4), and

$$\frac{d\sigma}{d\Omega} = r_0^2 \sin^2\theta, \tag{4.27}$$

where

$$r_0 \equiv \frac{e^2}{4\pi\epsilon_0 mc^2}. \tag{4.28}$$

The quantity r_0 is called the *classical radius of the electron*, and has the value $r_0 = 2.82 \times 10^{-11}$ m. Equations (4.27) and (4.28) describe *Thomson scattering* or *electron scattering*. The total cross section for scattering is obtained by integrating (4.27) over solid angle:

$$\sigma_T = \int \frac{d\sigma}{d\Omega} d\Omega = \frac{8\pi}{3} r_0^2. \tag{4.29}$$

This quantity is called the *Thomson cross section*, and has a value $\sigma_T = 6.65 \times 10^{-29}$ m^2.

Following our previous findings, the scattered radiation is linearly polarised in the xz plane.

The results obtained above are valid for a plane polarised incident electromagnetic wave. What if the incident wave is unpolarised? In that case we can assume without loss of generality that the incident unpolarised wave is propagating in the x direction and that scattering occurs in the xz plane. It is possible to decompose the electric field of the incident wave into components parallel to the z and y axes, $\mathbf{E} = E_y\widehat{\mathbf{y}} + E_z\widehat{\mathbf{z}}$. The component $E_z\widehat{\mathbf{z}}$ is at an angle θ to the scattering direction, and the component $E_y\widehat{\mathbf{y}}$ is at an angle $\frac{\pi}{2}$ to the scattering direction. The differential cross section is then the average of the contributions for scattering of linearly polarised radiation through angles θ and $\frac{\pi}{2}$:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{unpol}} = \frac{1}{2} r_0^2 \left(1 + \sin^2\theta\right). \tag{4.30}$$

There is an interesting distinction between our current results and a result found in Chapter 3. In the present context we find that Thomson scattering is independent of frequency: Equation (4.24) does not depend on ω. In Chapter 3, however, we investigated radiation from an oscillating dipole and found that there is a strong dependence of radiated power on frequency, according to Equation (3.35). This result was stated to be physically very important: for example it explains the blue colour of the sky. The resolution of this difference in behaviour is that in the oscillating dipole of Chapter 3 the amplitude of oscillation of the particles producing the radiation was fixed. Hence a higher frequency implied a greater acceleration, and according to (4.7) a greater power output. In the present context the electron producing the scattering oscillates freely in response to an incident wave. If the incident wave has a high frequency the amplitude of oscillation will be smaller, and the acceleration is independent of frequency. Hence the emitted power is also independent of frequency, according to (4.7). The two kinds of behaviour may be related by considering scattering by a harmonic oscillator, i.e. a particle assumed to be bound to a centre of force and driven by an incident wave with a frequency ω. At high frequencies of the incident wave the cross section for scattering approaches the Thomson value σ_T, because the energy associated with the incident wave is much greater than the 'binding energy' of the system, and the particle is nearly free. At low frequencies the cross section behaves according to

$$\sigma(\omega) \to \sigma_T \left(\frac{\omega}{\omega_0}\right)^4, \tag{4.31}$$

where ω_0 is the natural frequency of oscillation of the system. This regime is called *Rayleigh scattering*, and reproduces the $\sim \omega^4$ behaviour found earlier. In this case the incident wave is perturbing a bound particle. Rayleigh scattering is appropriate to describe scattering of sunlight by atmospheric dipoles, since the charges in the atmospheric dipoles are bound within molecules. (For a more complete of scattering by a harmonic oscillator, the reader is referred to Rybicki and Lightman.)

An astrophysical example of Thomson scattering is the appearance of the K corona of the Sun. During solar eclipses the corona — the tenous hot outer atmosphere of the Sun — is visible in white light up to several solar radii away from the limb of the Sun. One component of the observed light comes from Thomson scattering of solar radiation into the line of sight by coronal electrons. The 'K' in the name stems from the German word for continuum: the observed light does not carry the characteristic absorption lines of the solar spectrum (the Fraunhofer lines). Thomson scattering produces emission at the same frequency as the incident radiation, and so it will mimic the source spectrum if the scattering charges are at rest. However, the corona is hot ($T \approx 2 \times 10^6\,\mathrm{K}$), and so the absorption lines are smeared out by the Doppler shifts introduced by the random velocity component of the electron along the line of sight. Another component of the coronal emission is produced by scattering from dust particles: the F corona. This component does show the Fraunhofer lines (hence the F in the name), because the dust is slowly moving. Returning to the K corona, the theory developed above suggests that it will be polarised perpendicular to the radial direction to the Sun, and this is confirmed by observation. The intensity of the observed emission provides a measure of the coronal electron number density, as shown in the Problem Set for this week. The observed structures in the K corona trace magnetic field lines leading out into interplanetary space.

Problem Set 4

1. An electron is dropped close to the Earth's surface and falls under the force of gravity alone. What fraction of the potential energy loss in the first second is radiated? (Assume $a = g$ and $PE = mgh$.)

2. Light with initial flux F_0 passes through a region in which there are n scatterers per unit volume with scattering cross section σ. Show that after travelling a distance x the flux is

$$F(x) = F_0 e^{-n\sigma x}$$

   As discussed in the notes, the K corona is sunlight scattered by free electrons. The apparent brightness of the K corona at one solar radius from the sun's limb is about 10^{-8} that of the Sun's disc. Estimate the free electron density near the Sun.

3. Show that for a plane monochromatic electromagnetic wave with electric field

$$\mathbf{E}(t) = E_0 \sin(\omega t - kx)\widehat{\mathbf{z}}$$

   the Poynting vector is

$$\widehat{\mathbf{S}} = \epsilon_0 c E(t)^2 \widehat{\mathbf{x}}.$$

   What is the time average of the Poynting vector?

Chapter 5

Electromagnetic Radiation III

References: Griffiths, Rybicki and Lightman

Frequently in astrophysics we encounter radiation from relativistic particles, i.e. particles with a velocity that is significant relative to c. In this chapter we consider the electrodynamics of radiation by relativistic particles. Specifically, we consider the magnitude of the power radiated and the direction in which power is radiated. We begin with a review of some results from relativity.

In the following we use the standard notation for the *Lorentz factor*:

$$\gamma(v) \equiv \left(1 - \frac{v^2}{c^2}\right)^{-1/2}. \tag{5.1}$$

5.1 Relativistic transformations

5.1.1 Lorentz transformation

Consider a frame K and a second frame K' moving with velocity v with respect to K, in the x direction. The constancy of the speed of light in both frames requires that coordinates in the two frames are related by the *Lorentz transformation*:

$$x' = \gamma(x - vt) \tag{5.2a}$$

$$y' = y \tag{5.2b}$$

$$z' = z \tag{5.2c}$$

$$t' = \gamma(t - vx/c^2), \tag{5.2d}$$

where $\gamma = \gamma(v)$. Note that coordinates transverse to the relative motion of the two frames are unaffected. The inverse transformation (i.e. the unprimed coordinates in terms of the primed coordinates) is obtained by replacing v by $-v$ in Equations (5.2).

5.1.2 Transformation of velocity

Suppose that a particle moves with a velocity $\mathbf{u}'$ as measured in K'. What is the velocity of the particle in K? The answer involves a simple application of (5.2):

$$u_x = \frac{dx}{dt} = \frac{\gamma(dx' + vdt')}{\gamma(dt' + vdx'/c^2)} = \frac{u'_x + v}{1 + vu'_x/c^2}. \tag{5.3}$$

Similarly for the other components of **u**:

$$u_y = \frac{u'_y}{\gamma(1 + vu'_x/c^2)}, \quad u_z = \frac{u'_z}{\gamma(1 + vu'_x/c^2)}. \tag{5.4}$$

These results may be rewritten in terms of the components of velocity parallel and perpendicular to the direction of the relative motion between the two frames:

$$u_\parallel = \frac{u'_\parallel + v}{1 + vu'_\parallel/c^2}, \quad u_\perp = \frac{u'_\perp}{\gamma(1 + vu'_\parallel/c^2)}. \tag{5.5}$$

How does the direction of the motion change between frames? Introduce the angle $\theta = \tan^{-1}(u_\perp/u_\parallel)$ as shown in Figure 5.1. Using the transformation (5.5) it follows that

$$\tan\theta = \frac{u'\sin\theta'}{\gamma(u'\cos\theta' + v)}, \tag{5.6}$$

which is called the *aberration formula*. If $u' = c$ then Equation (5.6) becomes

$$\tan\theta = \frac{\sin\theta'}{\gamma(\cos\theta' + v/c)} \tag{5.7}$$

and also

$$\cos\theta = \frac{\cos\theta' + v/c}{1 + (v/c)\cos\theta'}. \tag{5.8}$$

Equations (5.7) and (5.8) describe the *aberration of light*.

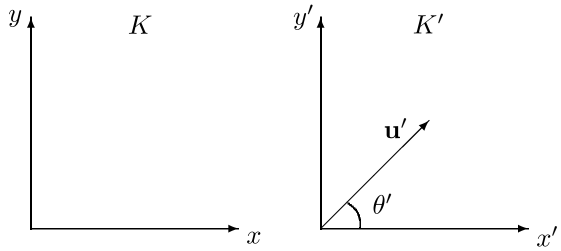


Figure 5.1: Transformation of direction of motion.

For the particular case that $\theta' = \frac{\pi}{2}$, it follows that

$$\sin\theta = \gamma^{-1}. \tag{5.9}$$

If γ is large then $\sin\theta$ and hence θ will be small:

$$\theta \approx \gamma^{-1} \quad (\gamma \gg 1). \tag{5.10}$$

This result has important implications for the direction of emission of radiation from relativitic particles. Assume that K' is the rest frame of a particle moving with a speed $v \approx c$ with respect to an observer in K. Assume the particle emits radiation isotropically. Photons emitted in the direction $\theta' = \frac{\pi}{2}$ travel in the direction $\theta \approx \gamma^{-1}$ in K. One half of the photons emitted in K' travel in the directions $|\theta| < \frac{\pi}{2}$. All of these photons travel in the narrow range of directions $|\theta| < \gamma^{-1}$ in the frame K. Hence radiation is concentrated in the direction of motion of the particle in K: very few photons are emitted with $|\theta| \gg \gamma^{-1}$. This effect is known as *beaming*, and is illustrated in Figure 5.2.

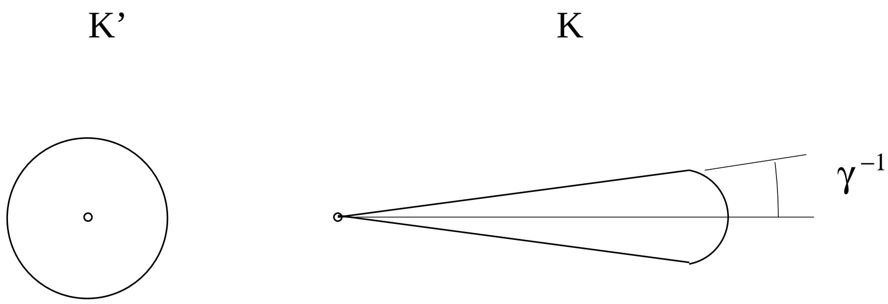


Figure 5.2: Relativistic beaming of radiation emitted isotropically in the rest frame K'.

5.1.3 Transformation of acceleration

It will also be useful to have at hand a result for the transformation of acceleration between reference frames. Noting that $a_x = du_x/dt$ it is straightforward to use (5.3), (5.4) and (5.2) to establish the transformation rules for acceleration. The results are more complicated than those for velocity, and we will only write down a special case that we will use shortly: when K' is instantaneously a rest frame, so that $u'_x = u'_y = u'_z = 0$. In that case

$$a_x = \gamma^{-3} a'_x, \quad a_y = \gamma^{-2} a'_y \quad \text{and} \quad a_z = \gamma^{-2} a'_z. \tag{5.11}$$

In terms of motion parallel and perpendicular to the relative velocity between the frames,

$$a'_{\parallel} = \gamma^3 a_{\parallel} \quad \text{and} \quad a'_{\perp} = \gamma^2 a_{\perp}. \tag{5.12}$$

5.1.4 Doppler effect

Suppose a clock at rest in K' measures a time interval $T' = t'_2 - t'_1$. The corresponding time interval in K is, according to (5.2)

$$T = t_2 - t_1 = \gamma(t'_2 - t'_1) = \gamma T', \tag{5.13}$$

which illustrates *time dilation.* Since $\gamma > 1$, periodic phenomena in K' have a longer period in K.

Time dilation is not dependent on (and does not include) the effects of the propagation time of light to an observer. Consider a source moving with a velocity v in an observer's rest frame. Suppose the source emits with a frequency ω' in its own rest frame. What will be the frequency of the source to a distant observer in K, measured on the basis of the arrival time of pulses? Figure 5.3 illustrates the problem.

We assume the source moves from position 1 to position 2 and emits one period of radiation in that time. The time taken to move from 1 to 2 is, according to the distant observer,

$$\Delta t = \frac{2\pi}{\omega'}\gamma, \tag{5.14}$$

using (5.13). The difference in arrival times of the photons from 1 and 2 is Δt minus the additional time for the photon from 1 to reach the observer:

$$\Delta t_A = \Delta t - d/c = \Delta t \left(1 - \frac{v}{c}\cos\theta\right). \tag{5.15}$$

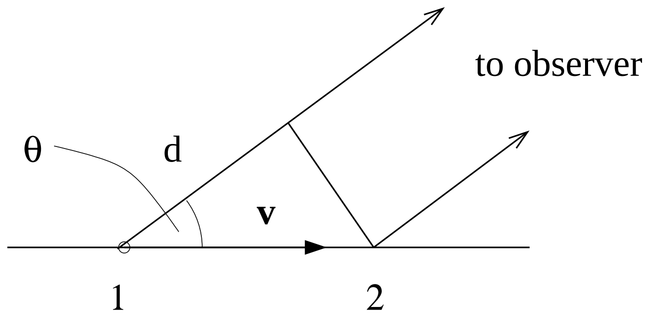


Figure 5.3: Geometry for Doppler effect.

The inferred frequency is then

$$\omega = \frac{2\pi}{\Delta t_A} = \frac{\omega'}{\gamma\left[1 - (v/c)\cos\theta\right]}, \tag{5.16}$$

which is the *relativistic Doppler formula*. The factor γ is the relativistic part of this formula: the factor $1 - (v/c)\cos\theta$ appears in the classical version as well, since it arises from a purely geometric consideration.

5.1.5 Transformation of energy and momentum

In special relativity the energy and momentum of a particle with velocity $\mathbf{u}$ in the frame K are given by

$$E = \gamma(u)m_0c^2 \quad \text{and} \quad \mathbf{p} = \gamma(u)m_0\mathbf{u}, \tag{5.17}$$

so in the frame K' we have

$$E' = \gamma(u')m_0c^2 \quad \text{and} \quad \mathbf{p}' = \gamma(u')m_0\mathbf{u}'. \tag{5.18}$$

It is straightforward to use the transformations established above to rewrite (5.18) as

$$E' = \gamma(E - vp_x), \tag{5.19a}$$

$$p'_x = \gamma(p_x - vE/c^2), \tag{5.19b}$$

$$p'_y = p_y, \tag{5.19c}$$

and

$$p'_z = p_z. \tag{5.19d}$$

Also recall that for a photon the relativistic energy and momentum are related by $E = pc$.

5.2 Radiation from fast moving charges

5.2.1 Generalising the Larmor formula

We are now in a position to establish detailed results for radiation by relativistic particles. Before proceeding it is useful to recall the approximation made in Chapter 4 which results in some of the formulae there being nonrelativistic.

Once again the expression for the acceleration field produced by a point charge is

$$\mathbf{E}_{\rm rad} = \frac{q}{4\pi\epsilon_0}\frac{r}{(\mathbf{r}\cdot\mathbf{u})^3}\left[\mathbf{r}\times(\mathbf{u}\times\mathbf{a})\right], \tag{5.20}$$

where $\mathbf{r}$ is the vector from the retarded position of the source to the observer, $\mathbf{a}$ is the acceleration of the charge at the retarded time and $\mathbf{u} = c\widehat{\mathbf{r}} - \mathbf{v}$, where $\mathbf{v}$ is the velocity at the retarded time. [1] In Chapter 4 we evaluated the associated Poynting flux,

$$\mathbf{S}_{\rm rad} = \frac{1}{\mu_0 c}E^2_{\rm rad}\widehat{\mathbf{r}}. \tag{5.21}$$

by making the assumption that the charge was at rest at the retarded time, so that $\mathbf{v} = \mathbf{0}$, and $\mathbf{u} = c\widehat{\mathbf{r}}$. This is a reasonable approximation provided $v \ll c$, but is strictly invalid for relativistic particles. The result was the Larmor formula.

Although we cannot assume $\mathbf{v} = \mathbf{0}$ for all times, it is always possible to transform into a frame K' in which the particle is *instantaneously* at rest. The particle does not remain at rest in this frame because it is accelerating, but for infinitesimal neighbouring times the particle is moving nonrelativistically, and so we can calculate the radiation emitted in the frame K' at the chosen instant using the Larmor formula, and then transform back to an observer's frame K.

Suppose that a total amount of energy dW' is emitted in the instantaneous rest frame K' in a time dt'. If we assume that the particle emits symmetrically in K', then total momentum of this radiation is zero, $\mathbf{dp'} = \mathbf{0}$. Using the transformation of energy relation (5.19a), we can obtain the total energy emitted in the frame K:

$$dW = \gamma dW'. \tag{5.22}$$

Also we have from the time dilation formula $dt = \gamma dt'$, and hence

$$P = \frac{dW}{dt} = \frac{dW'}{dt'} = P'. \tag{5.23}$$

Hence the total power emitted is independent of the frame (*invariant*) for any emitter that emits symmetrically in its instantaneous rest frame. From the Larmor formula (4.7) we have

$$P = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{q^2(a')^2}{c^3}. \tag{5.24}$$

Equations (5.12) allow us to express the accelerations in the instantaneous rest frame K' in (5.24) in terms of those in K:

$$\begin{aligned} P &= \frac{1}{4\pi\epsilon_0}\frac{2q^2}{3c^3}\left[(a'_{\parallel})^2 + (a'_{\perp})^2\right] \\ &= \frac{1}{4\pi\epsilon_0}\frac{2q^2}{3c^3}\gamma^4(a^2_{\perp} + \gamma^2 a^2_{\parallel}). \end{aligned} \tag{5.25}$$

It is a straightforward exercise to show that this can be rewritten in a vector form:

$$P = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{q^2}{c^3}\gamma^6\left[a^2 - \left|\frac{\mathbf{v}}{c}\times\mathbf{a}\right|^2\right]. \tag{5.26}$$

Equations (5.25) and (5.26) express the required relativistic generalisation of the Larmor formula.

[1] The parameter $\mathbf{u}$ has a specific definition and should not be confused with the symbol for arbitrary velocities used in earlier sections of this Chapter.

5.2.2 Directionality of emission

What about the directionality of the emitted radiation, i.e. the counterpart to Equation (4.6)? We expect from our earlier discussion of beaming that this should change drastically for highly relativistic particles.

Consider an amount of energy dW' emitted in the frame K' into the solid angle $d\Omega' = -d\mu' d\phi'$, where $\mu' = \cos\theta'$. There is a corresponding amount of momentum $\mathbf{dp}'$ associated with this radiation, where $dp' = dW'/c$. Applying Equation (5.19a) the corresponding amount of energy emitted into the solid angle $d\Omega = -d\mu d\phi$ in the frame K is

$$dW = \gamma(dW' + v dp'_x) = \gamma(1 + \beta\mu')dW', \tag{5.27}$$

where $\beta \equiv v/c$.

Next consider how the solid angle into which the radiation is emitted transforms. From (5.8) we have

$$\mu = \frac{\mu' + \beta}{1 + \beta\mu'}. \tag{5.28}$$

Differentiating leads to

$$d\mu = \frac{d\mu'}{\gamma^2(1 + \beta\mu')^2}. \tag{5.29}$$

Distances and hence angles perpendicular to the direction of relative motion between the two frames are unchanged, so $d\phi' = d\phi$. Hence we have

$$d\Omega = \frac{d\Omega'}{\gamma^2(1 + \beta\mu')^2}. \tag{5.30}$$

Putting Equations (5.27) and (5.30) together, the energy emitted per unit solid angle is

$$\frac{dW}{d\Omega} = \gamma^3(1 + \beta\mu')^3 \frac{dW'}{d\Omega'}. \tag{5.31}$$

To obtain the the power emitted per unit solid angle, we need to divide by a time interval. There are two possible choices:

1. $dt = \gamma dt'$. This is the time interval during which *emission* occurs in K. We will write $dP_e = dW/dt$.

2. $dt_A = \gamma dt'(1 - \beta\mu)$. According to (5.15) this is the time interval for radiation *received* by an observer in K, and includes a factor arising from the time light takes to reach the observer. We write $dP_r = dW/dt_A$.

With these two choices we obtain

$$\frac{dP_e}{d\Omega} = \frac{1}{\gamma^4(1 - \beta\mu)^3} \frac{dP'}{d\Omega'} \tag{5.32a}$$

and

$$\frac{dP_r}{d\Omega} = \frac{1}{\gamma^4(1 - \beta\mu)^4} \frac{dP'}{d\Omega'}, \tag{5.32b}$$

where we have used the relation

$$1 + \beta\mu' = \frac{1}{\gamma^2} \frac{1}{1 - \beta\mu}, \tag{5.33}$$

which follows from (5.28). The quantity $dP_r/d\Omega$ is the quantity that would be measured in astrophysical observations, so we will deal only with that from this point, and will drop the subscript r. However, it should be noted that some books (e.g. Griffiths) deal with $dP_e/d\Omega$.

The power radiated per unit solid angle in the instantaneous rest frame of the particle is given by Equation (4.6):

$$\frac{dP'}{d\Omega'} = \frac{q^2(a')^2 \sin^2 \Theta'}{16\pi^2\epsilon_0 c^3}, \tag{5.34}$$

where Θ' is the angle between $\mathbf{a}'$ and the direction of emission in K'. Writing $\mathbf{a}' = \mathbf{a}'_\parallel + \mathbf{a}'_\perp$ and combining (5.32b) and (5.34) we have

$$\frac{dP}{d\Omega} = \frac{q^2}{16\pi^2\epsilon_0 c^3} \frac{(\gamma^2 a_\parallel^2 + a_\perp^2)}{(1-\beta\mu)^4} \sin^2 \Theta', \tag{5.35}$$

where we have used (5.12). To evaluate Equation (5.35) we need to relate Θ' to angles in the observer's frame K. Figure 5.4 illustrates the geometry of the general case in frame K'. The velocity $\mathbf{v}$ defines the direction of transformation between frames and is taken to lie along the z axis. Without loss of generality we can assume that $\mathbf{a}'$ lies in the xz plane, and then the relevant angles are as shown. The general case is complicated, so we consider two simple cases of relevance in astrophysics.

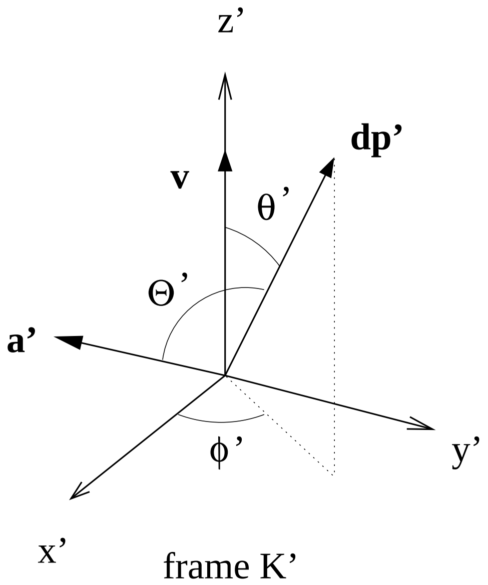


Figure 5.4: Geometry in the instantaneous rest frame K'.

Parallel velocity and acceleration

In this case $\Theta' = \theta'$ (see Figure 5.4) so

$$\sin^2 \Theta' = \sin^2 \theta' = \frac{\sin^2 \theta}{\gamma^2(1-\beta\mu)^2}, \tag{5.36}$$

where we have used (5.28). Also $a_\perp = 0$, so Equation (5.35) becomes

$$\frac{dP_\parallel}{d\Omega} = \frac{q^2}{16\pi^2\epsilon_0 c^3} a_\parallel^2 \frac{\sin^2 \theta}{(1-\beta\mu)^6}. \tag{5.37}$$

In the extreme relativistic case the angular distribution of radiation is as shown in Figure 5.5. The two lobes of emission familiar from the nonrelativistic case (Figure 4.1) are bent into the forward direction and stretched, as expected from the earlier arguments for beaming. The opening angle of the two lobes tends to the angle $\theta_{\rm max} \sim \gamma^{-1}$.

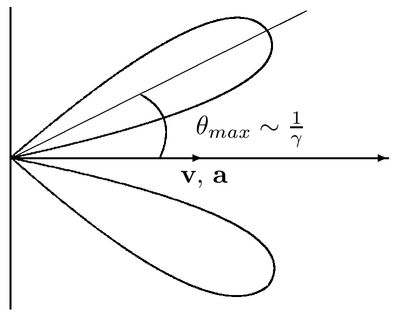


Figure 5.5: Radiation pattern for the extreme relativistic case when $\mathbf{v} \parallel \mathbf{a}$.

An example of the case $\mathbf{a} \parallel \mathbf{v}$ is when a high speed electron hits a metal target and rapidly decelerates, producing *bremsstrahlung*, or braking radiation. Since the incident particle is energetic and does not come from a thermal population, we will refer to this as *non-thermal bremsstrahlung*. This is the standard procedure for producing X-rays. The same process occurs in astrophysical situations. For example, during solar flares electrons are accelerated to mildly relativistic energies (10-100 keV) in closed magnetic structures in the solar corona ('coronal loops'). The electrons are constrained to move along magnetic field lines because of the Lorentz force, and so they follow magnetic fieldlines within the loops down to the lower levels of the solar atmosphere (the chromosphere). The density of gas is much higher in the chromosphere and so the energetic electrons are rapidly braked by Coulomb collisions with ambient ions. This produces bremsstrahlung radiation that comprises part of the X-ray emission of a solar flare. Figure 5.6 shows an example. A number of coronal loops involved in a flare (the dark structures) are shown, imaged in low energy X-rays that are produced in the corona (by *thermal bremsstrahlung*, a mechanism which will be discussed in Chapter 6). The contours show the position of higher energy X-rays produced by non-thermal bremsstrahlung of energetic electrons colliding with the denser atmosphere at the bottom of the loops. The high energy X-ray emission is observed to coincide with the 'footpoints' of the coronal loops.

Perpendicular velocity and acceleration

By reference to Figure 5.4, in this case $\cos\Theta' = \sin\theta'\cos\phi'$, so

$$\sin^2\Theta' = 1 - \sin^2\theta'\cos^2\phi' = 1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\mu)^2}, \tag{5.38}$$

using (5.36) and the fact that $\phi' = \phi$. Substituting this expression into (5.35) with $a_\parallel = 0$ gives the required result:

$$\frac{dP_\perp}{d\Omega} = \frac{q^2}{16\pi^2\epsilon_0 c^3}\frac{a_\perp^2}{(1-\beta\mu)^4}\left[1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\mu)^2}\right]. \tag{5.39}$$

In the extreme relativistic case the radiation pattern is as shown in Figure 5.7. In this case the radiation has a a dominant lobe in the direction of motion, and the other lobe is wrapped around into two lobes. Most of the radiation is again within a forward

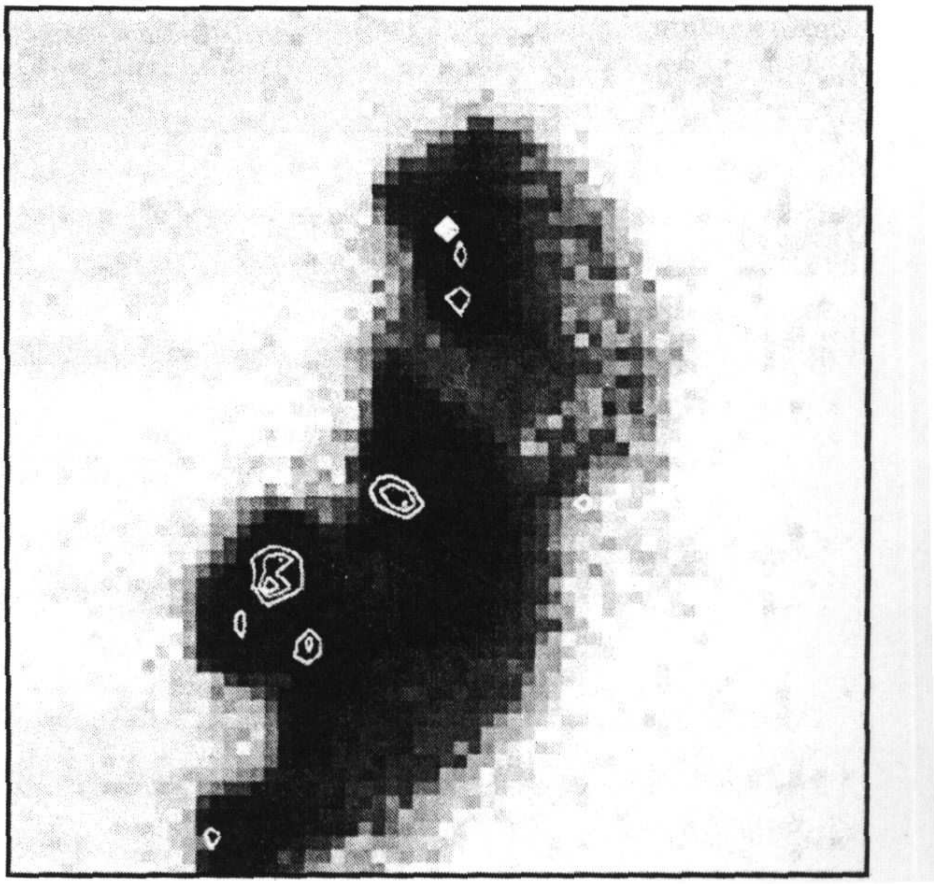


Figure 5.6: High energy X-ray emission (contours) at the footpoints of solar coronal loops. (From Hudson, H. and Ryan, J. 1995, Annual Reviews of Astronomy & Astrophysics **33**, 239.)

angle less than $\theta = \gamma^{-1}$. In this case ϕ appears in the expression — the pattern is not rotationally symmetric about the direction of motion.

An astrophysical example of relativistic motion with $\mathbf{a} \perp \mathbf{v}$ is a relativistic electron spiralling around a magnetic field line. An individual electron radiates in the forward direction as it rotates and so emits like a small lighthouse. The radiation produced is called *synchrotron radiation*, which is the mechanism underlying a variety of astrophysical radio sources including the Crab nebula. Synchrotron emission will be discussed in more detail in Chapter 6.

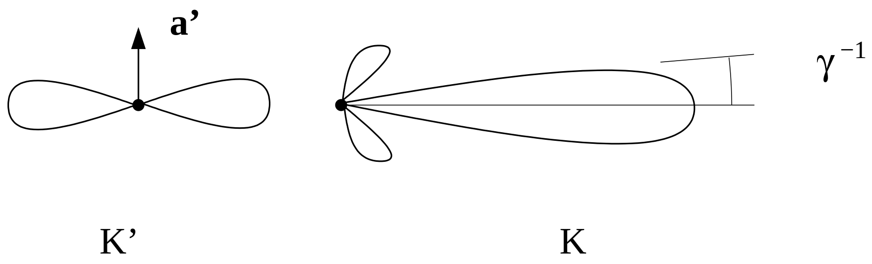


Figure 5.7: Radiation in the extreme relativistic case when $\mathbf{a} \perp \mathbf{v}$.

Problem Set 5

1. An astronomical object travels with velocity v at an angle of θ to the line of sight of a distant observer. Show that the apparent transverse velocity is:

$$v_{\rm app} = \frac{v \sin\theta}{1 - (v/c)\cos\theta}.$$

For a fixed v, find the angle at which $v_{\rm app}$ is a maximum. Hence show that $v_{\rm app}$ has a maximum value of γv, which can exceed c. Superluminal apparent velocities are observed with certain extragalactic radio sources!

2. Bradley (1728) observed the aberration of light by which stars appear to be displaced in the sky because of the motion of the Earth. A telescope must be directed away from the vertical by a maximum of $20.5''$ to observe stars that would be directly overhead to a stationary observer. What value for the radius of the Earth's orbit does this suggest ?

Chapter 6

Astrophysical Radiation

References: Rybicki and Lightman, Griffiths

In this chapter we will consider a number of astrophysical radiation mechanisms. The treatment will differ from the previous chapters in that we will not derive all of the results presented, because the derivations tend to be involved. However, the details can be found in the references.

6.1 Bremsstrahlung or free-free emission

We have already mentioned Bremsstrahlung, or braking radiation. In the astrophysical literature bremsstrahlung is often referred to as *free-free* emission since it arises from accelerations in collisions between unbound particles.

Collisions of like particles do not produce bremsstrahlung, since in this case the dipole moment $\sum q_i \mathbf{r}_i$ is proportional to the centre of mass $\sum m_i \mathbf{r}_i$, which is a constant of the motion. In electron-ion bremsstrahlung the electrons are the dominant radiators, because their smaller inertia results in larger accelerations. Hence in the following we consider only radiation from electrons in electron-ion collisions.

A full description of bremsstrahlung requires a quantum treatment. Chapters 3, 4 and 5 presented the classical theory of radiation emission. This theory becomes invalid, for example, when the frequency ν of radiation is comparable to the energy of the emitting particle. In quantum physics radiation consists of photons with energies $h\nu$, and if the emitting particle does not have this energy it cannot emit the photon. The classical theory also becomes inaccurate for close encounters between particles, since in quantum mechanics there are no exact trajectories. In the case of bremsstrahlung, a classical treatment produces formulae that have the correct functional dependence for most values of the relevant values of the parameters, so we will begin by discussing a classical model.

Emission by single speed electrons

Consider an electron moving in the fixed Coulomb field of a massive ion. We will assume the motion of the electron can be approximated by a straight line, i.e. we consider *small angle scattering*. Figure 6.1 illustrates the geometry of the situation. The distance of closest approach b is the *impact parameter* for the collision.

The dipole moment of the electron is $\mathbf{p} = -e\mathbf{R}$, so $\ddot{\mathbf{p}} = -e\dot{\mathbf{v}}$. Taking the Fourier

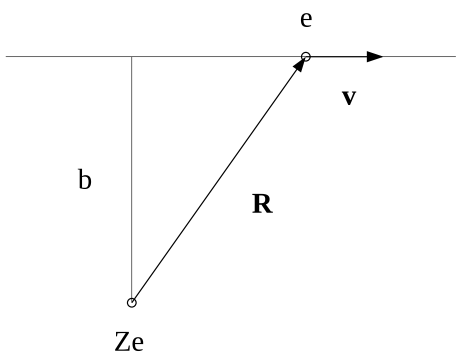


Figure 6.1: Geometry for small angle scattering of a single electron by an ion.

transform of this relation gives

$$-\omega^2\widetilde{\mathbf{p}}(\omega) = -\frac{e}{2\pi}\int_{-\infty}^{\infty}\dot{\mathbf{v}}e^{i\omega t}dt. \tag{6.1}$$

A characteristic time for the collision is given by the ratio of the impact parameter and the velocity, $\tau = b/v$. If $\omega\tau \gg 1$ then the argument of the exponential in (6.1) undergoes many cycles during the collision, and the resulting value of the integral will be small. If $\omega\tau \ll 1$ then the exponential in (6.1) will be approximately unity. Hence we can make the crude approximation

$$\widetilde{\mathbf{p}}(\omega) = \begin{cases} e\Delta\mathbf{v}/(2\pi\omega^2) & \text{if } \omega\tau \ll 1 \\ 0 & \text{if } \omega\tau \gg 1, \end{cases} \tag{6.2}$$

where $\Delta\mathbf{v}$ is the change in velocity during the collision.

Substituting (6.2) into Equation (4.21) gives an estimate of the total power radiated by the particle per unit frequency during its interaction with the ion:

$$\frac{dW}{d\omega} = \begin{cases} e^2(\Delta v)^2/(6\pi^2\epsilon_0 c^3) & \text{if } \omega\tau \ll 1 \\ 0 & \text{if } \omega\tau \gg 1. \end{cases} \tag{6.3}$$

To evaluate (6.3) we need to estimate Δv. The velocity parallel to the straight line path will be the same at large times after the collision as it was at large times before, by symmetry. Hence we need only consider the change in the component of velocity perpendicular to the path. Integrating the component of acceleration perpendicular to the path over all times gives

$$\Delta v = \frac{Ze^2}{4\pi\epsilon_0 m}\int_{-\infty}^{\infty}\frac{bdt}{(b^2+v^2t^2)^{3/2}}. \tag{6.4}$$

Assuming the velocity is constant along the path the integral can be evaluated, giving

$$\Delta v = \frac{Ze^2}{2\pi\epsilon_0 mbv}, \tag{6.5}$$

and hence (4.21) becomes

$$\frac{dW(b,v)}{d\omega} = \frac{8}{3}\frac{Z^2e^6}{(4\pi\epsilon_0)^3\pi m^2c^3b^2v^2} \quad \text{for } b \ll v/\omega, \tag{6.6}$$

and zero for $b \gg v/\omega$.

Equation (6.6) is the result for a single electron-ion collision. Consider the situation that an electron moves with speed v in an ionised gas (*plasma*) where the number density of ions is n_i. In time dt the electron will interact with $n_i\,(2\pi b\,db)\,(v\,dt)$ ions with impact parameter between b and $b+db$, as shown in Figure 6.2. In a small volume of plasma dV there are $n_e\,dV$ electrons. Hence the total number of collisions with impact parameter in the chosen range in the chosen volume and in the chosen time is $(n_e\,dV)n_i\,(2\pi b\,db)\,(v\,dt)$. It follows that the total rate of power emitted per unit volume and per unit frequency range is

$$\frac{dP(v)}{d\omega dV} = n_e n_i 2\pi v \int_0^{\omega/v} b db \frac{dW(b,v)}{d\omega}, \tag{6.7}$$

where the upper limit to the integral is taken to be the extreme value of b identified earlier. To evaluate the integral in (6.7) using (6.6) we need to introduce a lower limit b_0 to the integral as well. The result is

$$\frac{dP(v)}{d\omega dV} = \frac{16 n_e n_i Z^2 e^6}{3(4\pi\epsilon_0)^3 m^2 c^3 v} \ln\left(\frac{v}{b_0\omega}\right). \tag{6.8}$$

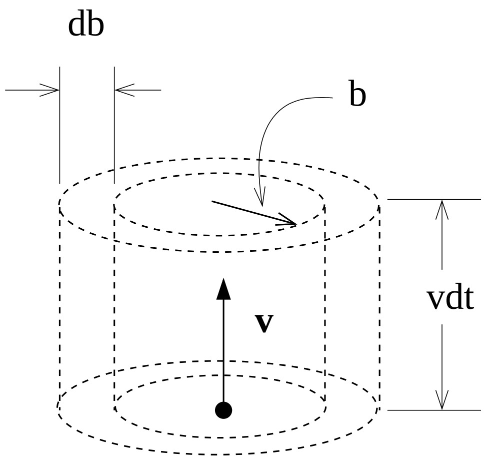


Figure 6.2: In time dt, a given electron interacts with all ions with impact parameters b to $b+db$ in the volume shown.

It remains to estimate b_0 on an ad hoc basis (e.g. it can be taken to be the value of b for which the straight line approximation is no longer valid). When this is done, Equation (6.8) is the classical result for small angle collisions. The quantum mechanical result, including large angle collisions, is

$$\frac{dP(v)}{d\omega dV} = \frac{16\pi n_e n_i Z^2 e^6}{3\sqrt{3}(4\pi\epsilon_0)^3 m^2 c^3 v} g_{ff}(v,\omega), \tag{6.9}$$

which differs from (6.8) in the replacement of the logarithm by the *Gaunt factor* $g_{ff}(v,\omega)$:

$$\ln\left(\frac{v}{b_0\omega}\right) \to \frac{\pi}{\sqrt{3}} g_{ff}(v,\omega). \tag{6.10}$$

The Gaunt factor is typically close to unity, and so the classical calculation is remarkably accurate. In part this is because the uncertain factors appear inside a logarithm.

Thermal bremsstrahlung

An important example of bremsstrahlung is *thermal bremsstrahlung*, where the emitting electrons come from a thermal population. To obtain the power emitted per unit volume in thermal bremsstrahlung we need to average the single speed formula (6.9) over a thermal distribution of speeds,

$$\mathrm{prob}(v)dv \propto v^2 e^{-mv^2/(2k_BT)}dv, \tag{6.11}$$

i.e. we need to evaluate

$$\frac{dP_{\mathrm{th}}}{dVd\omega} = \frac{4}{\pi^{1/2}}\left(\frac{m}{2k_BT}\right)^{3/2}\int_{v_{\min}}^{\infty}\frac{dP(v)}{dVd\omega}v^2 e^{-mv^2/(2k_BT)}dv, \tag{6.12}$$

where the factors out the front arise from the normalisation of (6.11). The lower limit $v_{\min}$ in the integral arises because an electron must have sufficient energy to emit a photon:

$$\frac{1}{2}mv_{\min}^2 = \hbar\omega. \tag{6.13}$$

Evaluating the integral and writing the result in terms of frequency $\nu = \omega/(2\pi)$ leads to

$$\frac{dP_{\mathrm{th}}}{dVd\nu} = An_en_iT^{-1/2}e^{-h\nu/(k_BT)}, \tag{6.14}$$

where

$$A = \left(\frac{m}{6\pi k_B}\right)^{1/2}\frac{n_en_iZ^2e^6}{3\pi\epsilon_0^3m^2c^3}\overline{g_{ff}} \approx 6.8\times10^{-39}\overline{g_{ff}}\,\mathrm{W\,m^3K^{1/2}Hz^{-1}}. \tag{6.15}$$

The factor $\overline{g_{ff}}$ is a velocity-averaged Gaunt factor. Integrating (6.14) over all frequencies leads to

$$\frac{dP_{\mathrm{th}}}{dV} = A'n_en_iT^{1/2}, \tag{6.16}$$

where

$$A' = \left(\frac{mk_B}{6\pi}\right)^{1/2}\frac{n_en_iZ^2e^6}{3\pi\epsilon_0^3m^2c^3}\overline{g_B} \approx 1.4\times10^{-28}\overline{g_B}\,\mathrm{W\,m^3\,K^{-1/2}\,Hz}, \tag{6.17}$$

where $\overline{g_B} \approx 1.2$ is a frequency average of the velocity-averaged Gaunt factors.

In the following we consider optically thin thermal bremsstrahlung sources, which are common in astrophysics. (The absorption of photons by free electrons becomes important when source densities are large, and for lower energy photons.) For an optically thin source at a single temperature, the observed emission follows (6.14) and (6.16), and on a log log plot the the spectrum is fairly flat below a cutoff at $\nu \approx k_BT/h$. Emission from a volume element ΔV is proportional to $n_en_i\Delta V$. Generally this is equal to $n_e^2\Delta V$, which is called the *emission measure* of the source element. Identification of a spectrum as thermal bremsstrahlung leads to two physical parameters of the source: the temperature and the total emission measure, $\int n_e^2 dV$.

It is also worthwhile to note that thermal bremsstrahlung is unpolarised, since it arises from accelerations of electrons in random directions.

An example of thermal bremsstrahlung is the soft X-ray emission from the Sun's corona, which is a plasma at about $T = 2\times10^6\,\mathrm{K}$. Regions of hotter plasma are produced in the corona during solar flares, as shown in Figure 6.3. The figure shows a sequence of spectra in the X-ray range (10-100 keV) observed during a flare. The spectra are at first power laws, indicating a non-thermal population of emitting electrons. Later in the event a thermal component appears as a very steep spectrum below about 30 keV (what is observed is essentially the $\exp(-E/k_BT)$ part of the thermal bremsstrahlung spectrum). Fitting indicates that the thermal source has a temperature of a few times 10^7 K, and so this emission has been called the *superhot* component. The standard interpretation of the observations is that a power-law spectrum of accelerated electrons is produced in the corona during a flare. These particles precipitate to the lower levels of the atmosphere and produce non-thermal bremsstrahlung there (see Figure 5.6), which accounts for the power law spectra in Figure 6.3. The braking of the electrons in the lower atmosphere also produces intense heating, which is believed to lead to ejection of hot dense material back into the corona. Some of that material produces the thermal component seen in Figure 6.3.

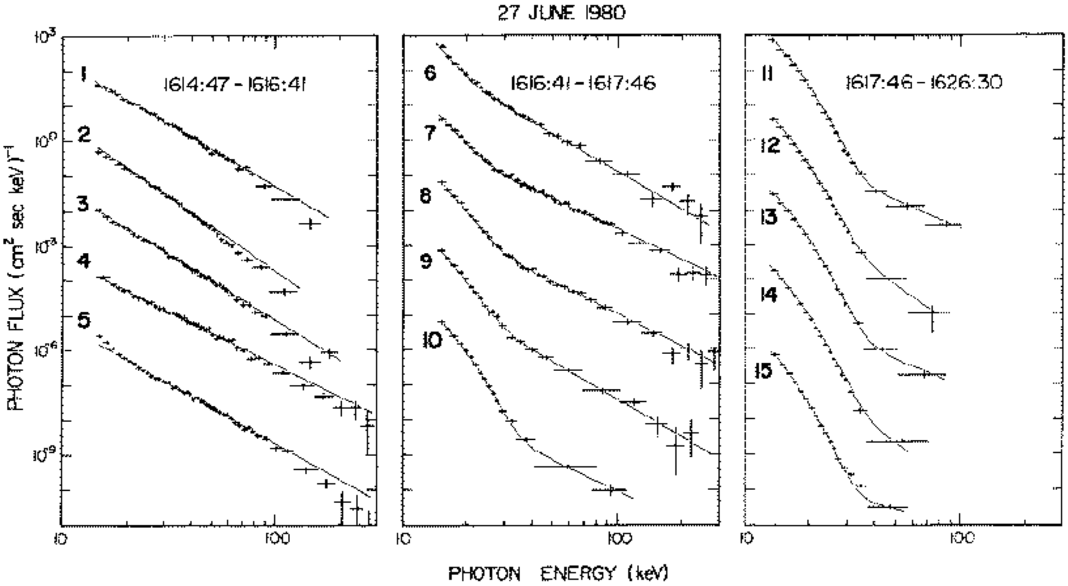


Figure 6.3: Sequence of X-ray spectra during a solar flare (From Lin, R., Schwartz, R., Pelling, R. and Hurley, K. 1981, Astrophysical Journal Letters **251**, L109.

6.2 Synchrotron Radiation

Particles spiralling around magnetic field lines are accelerating and hence radiate. If the particle is nonrelativistic the emission is called *cyclotron radiation*, and occurs at the frequency of gyration of the particle. For extreme relativistic particles the emission is more complicated because of the beaming effect, and the frequency of emission is broad, extending to many times the gyration frequency: this is *synchrotron radiation*, as mentioned in Chapter 5.

Total power in synchrotron radiation

The relativistic equation of motion of a particle with rest mass m_0 and charge q subject only to the Lorentz force due to a magnetic field $\mathbf{B}$ is

$$\frac{d}{dt}(\gamma m_0 \mathbf{v}) = q\mathbf{v} \times \mathbf{B}. \tag{6.18}$$

The energy of a particle is unchanged by a magnetic field, and so γ and v are constants. Separating the motion into components parallel and perpendicular to the field, we have

$$\frac{d\mathbf{v}_\parallel}{dt} = 0, \quad \frac{d\mathbf{v}_\perp}{dt} = \frac{q}{\gamma m_0}\mathbf{v}_\perp \times \mathbf{B} \tag{6.19}$$

It follows from $v = \text{const}$ and the first of (6.19) that $v_\perp$ is a constant. The second of (6.19) indicates that the particle is subject to a constant acceleration $a_\perp = qv_\perp B/(\gamma m_0)$ perpendicular to $\mathbf{v}$. Hence the motion perpendicular to $\mathbf{B}$ is circular motion. The first of (6.19) indicates that the motion parallel to the field is unaffected by the field, and so in general the particle undergoes helical motion. The frequency of gyration is

$$\omega_B = \frac{a_\perp}{v_\perp} = \frac{qB}{\gamma m_0}. \tag{6.20}$$

Applying the relativistic generalisation of the Larmor formula (5.25) with $a_\perp = \omega_B v_\perp$ and assuming the particle is an electron, the power radiated by the particle is

$$P = \frac{8}{3}\pi\epsilon_0 r_0^2 c\gamma^2 v_\perp^2 B^2, \tag{6.21}$$

where r_0 is the classical radius if the electron, defined by (4.28). This formula applies for a certain value of $v_\perp = v\sin\alpha$, where α is the angle of inclination of the velocity of the particle to the magnetic field (the *pitch angle*). For an isotropic distribution of velocities it is necessary to average over pitch angle:

$$\langle v_\perp^2 \rangle = \frac{v^2}{4\pi}\int \sin^2\alpha\, d\Omega = \frac{2v^2}{3}, \tag{6.22}$$

which leads to

$$P = \frac{4}{3}\sigma_T \beta^2 c\gamma^2 U_B, \tag{6.23}$$

where $\sigma_T = 8\pi r_0^2/3$ is the Thomson cross section, $U_B = B^2/(2\mu_0)$ is the magnetic energy density, and $\beta = v/c$.

Spectrum of synchrotron radiation

As mentioned above, cyclotron emission occurs at the frequency of gyration of the electron. With synchrotron radiation, the beaming of radiation affects the observed spectrum. To understand why, consider Figure 6.4. As the electron gyrates it produces radiation in a narrow cone pointing in the direction of motion, with an opening angle $\Delta\theta \sim \gamma^{-1}$. A distant observer will see a pulse of emission from the electron when the cone sweeps through the line of sight. The electron performs one gyration in a time $T = 2\pi/\omega_B$. The time interval during which the cone of emission is directed towards the observer is of order $\Delta t = (\Delta\theta/2\pi)T = 1/(\gamma\omega_B)$. The time interval between the beginning and end of receipt of radiation by the observer is smaller than this by a factor $1-v/c$, since in the time Δt the electron has moved towards the observer, and so photons at the end of the pulse are emitted closer to the observer than those at the beginning (this is the geometrical factor in the Doppler effect identified in § 5.14). Hence the duration of the pulse received by the observer is of order $\Delta t_A = (\gamma\omega_B)^{-1}(1-v/c) \approx (2\gamma^3\omega_B)^{-1}$. A pulse of duration Δt contains Fourier components $\omega \lesssim (\Delta t)^{-1}$. Hence we expect synchrotron radiation to contain frequencies

$$\omega \lesssim \omega_c \equiv 2\gamma^3\omega_B. \tag{6.24}$$

This estimate applies strictly only to particles with a pitch angle $\alpha = \pi/2$. For particles with an arbitrary pitch angle ω_c is multiplied by $\sin\alpha$.

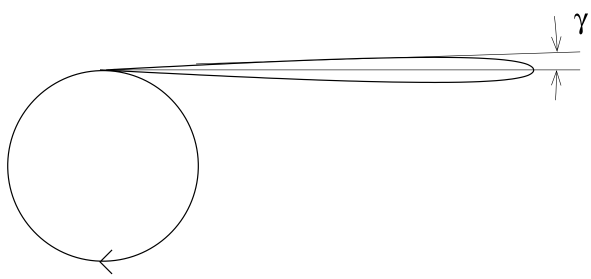


Figure 6.4: Synchrotron emission from a particle with $\gamma \gg 1$.

A detailed analysis (e.g. Rybicki and Lightman) indicates that synchrotron emission from a single electron consists of spikes at integer multiples of ω_B (harmonics), up to about ω_c. The power per unit frequency spectrum can be written in the form $P(\omega) = p(q, B, \alpha, m)F(\omega/\omega_c)$, where $F(\omega/\omega_c)$ describes the decay of the spectrum at large frequencies. An important result can be derived from this form alone. In astrophysics it is common to deal with emission from a power-law distribution of accelerated particles,

$$N(\gamma)d\gamma = C\gamma^{-p}d\gamma. \tag{6.25}$$

The number p is called the *index* of the power law. The synchrotron power from these particles is obtained by averaging the power per particle over the distribution of particle energies,

$$P_{\rm tot}(\omega) = C\int_{\gamma_1}^{\gamma_2} P(\omega)\gamma^{-p}d\gamma \propto \int_{\gamma_1}^{\gamma_2} F(\omega/\omega_c)\gamma^{-p}d\gamma, \tag{6.26}$$

where the limits in the integral represent limits to the particle spectrum. Changing the variable of integration to $x = \omega/\omega_c$ and noting that $\omega_c \propto \gamma^2$ (since $\omega_B \propto \gamma^{-1}$) leads to

$$P_{\rm tot}(\omega) \propto \omega^{-(p-1)/2}\int_{x_1}^{x_2} F(x)x^{(p-3)/2}dx. \tag{6.27}$$

The integral can be taken to be constant, and so we have established that a power law distribution of electrons with an index p in their energy distribution produces a power-law frequency spectrum of synchrotron radiation, with an index $s = (p-1)/2$.

The preceding discussion of synchrotron radiation has implicitly assumed optically thin sources. The synchrotron process also provides an absorption mechanism of importance in astrophysics, although we will not discuss it here. Finally, synchrotron radiation is highly polarised. For electrons with a pitch angle of $\pi/2$, the radiation is linearly polarised perpendicular to the magnetic field. In the more general case the radiation is elliptically polarised.

6.3 Compton scattering

When we treated Thomson scattering in § 4.2 we ignored the possibility of transfer of energy from the photons of the incident radiation to the scattering charge, i.e. we assumed that the scattered frequency was the same as the incident frequency. This assumption is invalid for high energy photons, e.g. X-ray and γ-ray photons, as we will see.

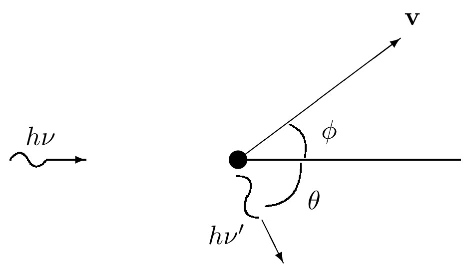


Figure 6.5: Compton Scattering.

Figure 6.5 shows a photon of wavelength λ scattering elastically from a particle initially at rest. Conservation of momentum in the vertical direction gives

$$p_e \sin\phi = (h\nu'/c)\sin\theta, \tag{6.28}$$

where p_e is the momentum of the electron after the collision, and conservation of momentum in the horizontal direction gives

$$h\nu/c = (h\nu'/c)\cos\theta + p_e\cos\phi. \tag{6.29}$$

Combining (6.28) and (6.29) to eliminate ϕ leads to

$$p_e^2c^2 = h^2\nu^2 - 2h^2\nu\nu'\cos\theta + h^2(\nu')^2. \tag{6.30}$$

Conservation of energy can be written

$$h\nu + m_0c^2 = h\nu' + \left[m_0c^4 + p_e^2c^2\right]^{1/2}, \tag{6.31}$$

where m_0 is the rest energy of the electron. Combining (6.30) and (6.31) to eliminate p_e and solving for $\lambda' = c/\nu'$ leads to

$$\lambda' = \lambda + \lambda_C(1-\cos\theta), \tag{6.32}$$

where

$$\lambda_C = \frac{h}{m_0c} \tag{6.33}$$

is the *Compton wavelength* of the scattering particle, which describes the characeristic change in wavelength of a scattering event with the particle. For electrons λ_C =2.4 pm. We see from this that scattering from electrons is adequately described by Thomson scattering at visible wavelengths, but this is not the case at γ-ray wavelengths.

It is possible to calculate a scattering cross section for Compton scattering. Quantum mechanics is required and the expression (the *Klein-Nishina* formula) is rather daunting. For the cases of low and high energy photons the result is simpler. Introducing the ratio

$$x = \frac{h\nu}{m_0c^2} \tag{6.34}$$

of the photon energy to the electron energy, the limiting forms of the Klein-Nishima cross section are

$$\sigma_C = \begin{cases} \sigma_T[1-2x+\frac{26}{5}x^2\ldots] & \text{for } x \ll 1 \\ \frac{3}{8}\sigma_T x^{-1}(\ln 2x + \frac{1}{2}) & \text{for } x \gg 1 \end{cases} \tag{6.35}$$

We see that the Thomson cross section is recovered in the classical limit.

As implied by its name, the *inverse Compton effect* is the normal effect in reverse. A highly energetic particle (usually an electron) collides with a low energy photon and produces a high energy photon. This process can be treated by Lorentz transforming to the rest frame of the electron, applying Equation (6.32), and then tranforming back to the observer's frame. Figure 6.6 illustrates the geometry of the process in the observer's frame K and in the electron rest frame K'. At this point we change notation, and denote the energy of the photon before the collision E, and label the energy after E_1.

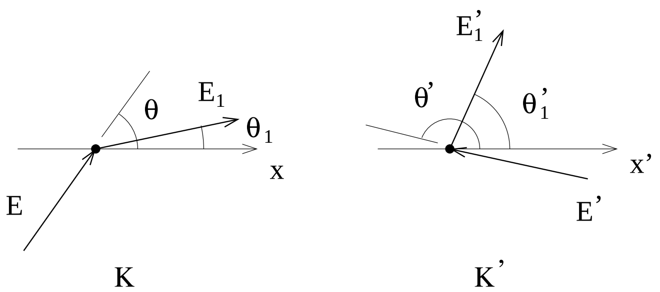


Figure 6.6: Geometry of inverse Compton scattering in an observer's frame K and in the electron rest frame K'.

From the energy transformation formulae (5.19a) we have

$$E' = \gamma E \left(1 - \frac{v}{c} \cos\theta\right) \tag{6.36}$$

and

$$E_1 = \gamma E_1' \left(1 + \frac{v}{c} \cos\theta_1'\right). \tag{6.37}$$

Applying (6.32) in the rest frame assuming $x \ll 1$ gives

$$E_1' \approx E' \left[1 - \frac{E'}{m_0 c^2} (1 - \cos\Theta)\right], \tag{6.38}$$

where

$$\cos\Theta = \cos\theta_1' \cos\theta' + \sin\theta' \sin\theta_1' \cos(\phi' - \phi_1'), \tag{6.39}$$

and where ϕ_1' and ϕ' are the azimuthal angles of the scattered and incident photon in the rest frame.

The important thing to note from these formulae is that $E_1 \sim \gamma^2 E$. The Lorentz transformation to the rest frame introduces a factor of γ, and the transformation back to the observer's frame also introduces a factor γ. This indicates that it is possible to produce photons of very large energy by inverse Compton scattering.

The inverse Compton effect is thought to account for X-ray emission from a variety of astrophysical sources. For example, certain X-ray sources are believed to be accreting black hole binaries that involve relativistic flows. These systems produce X-rays by inverse Compton scattering of ambient photons by electrons in the flows.

6.4 Transverse electromagnetic waves in a plasma

So far we have assumed that the medium through which astrophysical radiation propagates is a vacuum. Actually the medium is (in general) a low density plasma, and the motion of the charges in the plasma in response to a passing electromagnetic wave alters the transmitted wave.

To understand the effect, it is sufficient to consider plane wave solutions

$$\mathbf{B} = \widetilde{\mathbf{B}} \exp i(\mathbf{k}\cdot\mathbf{x} - \omega t), \quad \mathbf{E} = \widetilde{\mathbf{E}} \exp i(\mathbf{k}\cdot\mathbf{x} - \omega t) \tag{6.40}$$

to the Maxwell equations. With this assumption Equations (3.1) become:

$$\begin{aligned} i\mathbf{k}\cdot\widetilde{\mathbf{E}} &= \frac{\widetilde{\rho}}{\epsilon_0} \qquad & i\mathbf{k}\cdot\widetilde{\mathbf{B}} &= 0 \\ i\mathbf{k}\times\widetilde{\mathbf{E}} &= i\omega\widetilde{\mathbf{B}} \qquad & i\mathbf{k}\times\widetilde{\mathbf{B}} &= \mu_0\widetilde{\mathbf{J}} - \frac{i\omega}{c^2}\widetilde{\mathbf{E}}. \end{aligned} \tag{6.41}$$

The procedure of looking for plane wave solutions is equivalent to Fourier transforming the Maxwell equations in space and time.

Since electrons have much less inertia than ions, it is reasonable to consider only the motion of electrons in the plasma in response to a passing electromagnetic wave. The ratio of the magnetic force on an electron in the plasma to the electric force is

$$\frac{e|\mathbf{v}\times\mathbf{B}|}{e|\mathbf{E}|} \sim v\frac{B}{E} = \frac{v}{c}, \tag{6.42}$$

since for a transverse EM wave $E/B = c$. Hence for nonrelativistic motion of the electrons the electric force is much greater than the magnetic force, and the equation of motion of an electron becomes

$$m\dot{\mathbf{v}} = -e\mathbf{E}. \tag{6.43}$$

Substituting plane wave solutions in this equation leads to

$$\widetilde{\mathbf{v}} = \frac{e}{i\omega m}\widetilde{\mathbf{E}}. \tag{6.44}$$

The amplitude of the current density in the plasma as a result of the motion of the electrons is

$$\widetilde{\mathbf{J}} = -ne\widetilde{\mathbf{v}} = \sigma\widetilde{\mathbf{E}}, \tag{6.45}$$

where

$$\sigma = \frac{ine^2}{m\omega} \tag{6.46}$$

is the *conductivity* of the plasma.

Inserting plane wave solutions in the charge conservation equation (3.2) (which is equivalent to two of the Maxwell equations) gives

$$-i\omega\widetilde{\rho} + i\mathbf{k}\cdot\widetilde{\mathbf{J}} = 0. \tag{6.47}$$

Combining (6.45) and (6.47) gives

$$\widetilde{\rho} = \sigma\omega^{-1}\mathbf{k}\cdot\widetilde{\mathbf{E}}. \tag{6.48}$$

Equations (6.45) and (6.48) relate the source terms in the Maxwell equations back to the fields. Substituting these expressions for the sources back into the Maxwell equations leads to a source-free version of the Maxwell equations:

$$\begin{aligned} i\mathbf{k}\cdot\epsilon\widetilde{\mathbf{E}} &= 0 \qquad & i\mathbf{k}\cdot\widetilde{\mathbf{B}} &= 0 \\ i\mathbf{k}\times\widetilde{\mathbf{E}} &= i\omega\widetilde{\mathbf{B}} \qquad & i\mathbf{k}\times\widetilde{\mathbf{B}} &= -i\frac{\omega}{c^2}\epsilon\widetilde{\mathbf{E}}, \end{aligned} \tag{6.49}$$

where

$$\epsilon \equiv 1 - \frac{\sigma}{i\epsilon_0\omega} \tag{6.50}$$

is the dielectric constant of the medium. Using (6.46) we can rewrite the dielectric constant as

$$\epsilon = 1 - \left(\frac{\omega_p}{\omega}\right)^2, \tag{6.51}$$

where

$$\omega_p = \left(\frac{ne^2}{m\epsilon_0}\right)^{1/2} \tag{6.52}$$

is the *plasma frequency*. Numerically,

$$\omega_p = 56.4\sqrt{n}\,\mathrm{s}^{-1}. \tag{6.53}$$

The next step is to solve Equations (6.49). First note that these equations imply that $\mathbf{k}$, $\widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{B}}$ form a triad of orthogonal vectors. Taking $\mathbf{k}$ crossed with $\mathbf{k}\times\widetilde{\mathbf{E}}$ and using (6.49) leads to

$$\left(k^2 - \frac{\omega^2}{c^2}\epsilon\right)\widetilde{\mathbf{E}} = 0. \tag{6.54}$$

For a non-trivial solution ($\widetilde{\mathbf{E}} \neq 0$) we see that

$$k^2 = \frac{\omega^2}{c^2}\epsilon, \tag{6.55}$$

which can be rewritten using (6.51) as

$$\omega^2 = \omega_p^2 + k^2c^2. \tag{6.56}$$

Equation (6.56) is the *dispersion relation* for the propagation of transverse electromagnetic waves in the plasma. This equation implies that ω_p is the minimum value of the frequency of the waves: transverse EM waves cannot propagate below the plasma frequency. This fact has many important consequences. For example, it explains why AM radio waves are reflected by the Earth's ionosphere (the ionised upper layer of the atmosphere), but FM radio waves pass through. At some level in the ionosphere the plasma density is sufficiently high that the local plasma frequency is greater than the frequency of the AM radio wave, and at that level the wave is reflected (the energy of the wave has to go somewhere). However, FM waves have a higher frequency, which is always above the local plasma frequency in the ionosphere, and hence they are transmitted unimpeded. The reflection of low frequency radio waves can be used to probe the ionosphere from the ground. Measurement of the time delay for the return of pulses at different frequencies provides a means to determine the density of the ionosphere as a function of height.

Provided $\omega > \omega_p$, monochromatic waves propagate at the *phase velocity*

$$v_\phi \equiv \frac{\omega}{k} = \frac{c}{n_r}, \tag{6.57}$$

where

$$n_r \equiv \epsilon^{1/2} = \left[1 - \left(\frac{\omega_p}{\omega}\right)^2\right]^{1/2} \tag{6.58}$$

is the *refractive index* of the medium. The phase velocity of these waves is always greater than c. This is not a problem however, since an infinite monochromatic wave does not convey any information. To encode information it is necessary to modulate the wave, which involves sending a number of different frequencies. The envelope of any modulation (which conveys the information) travels with the *group velocity*

$$v_g \equiv \frac{\partial \omega}{\partial k} = n_r c, \tag{6.59}$$

which is always less than c.

The dependence of the phase and group speeds on frequency is called *dispersion*. An astrophysical application of dispersion concerns radio waves from *pulsars*, which are compact objects (believed to be rapidly rotating magnetised neutron stars) which produce pulsed radio emission. The standard interpretation of the emission is that a continuous beam of radio emission is produced by some means at the magnetic pole of the pulsar. The magnetic and rotation axes of the pulsar are not aligned, so the rotation of the pulsar causes the beam to sweep across the sky, leading to a pulse of emission at an observer when the beam crosses the line of sight to the observer.

Suppose a pulsar is at a distance d from Earth. The time for a pulse at a frequency ω to reach the Earth is

$$t_p = \int_0^d \frac{ds}{v_g}. \tag{6.60}$$

The plasma frequency in interstellar space is low ($\sim 10^3\,\mathrm{Hz}$), so we can expand the square root in the group velocity:

$$v_g^{-1} = \frac{1}{c}\left[1 - \left(\frac{\omega_p}{\omega}\right)^2\right]^{-1/2} \approx \frac{1}{c}\left[1 + \frac{1}{2}\left(\frac{\omega_p}{\omega}\right)^2\right], \tag{6.61}$$

leading to

$$t_p = \frac{d}{c} + \frac{1}{2c\omega^2}\int_0^d \omega_p^2 ds. \tag{6.62}$$

The first term in (6.62) is the free space transit time, and the second is the plasma correction. Based on observations of pulse arrival at different radio frequencies ω it is possible to measure the rate of change of arrival time with respect to frequency. Using (6.62) and the definition of plasma frequency we have

$$\frac{dt_p}{d\omega} = \frac{-e^2}{m\epsilon_0 c\omega^3}\mathcal{D}, \tag{6.63}$$

where

$$\mathcal{D} = \int_0^d n\, ds \tag{6.64}$$

is the *dispersion measure*. If an estimate of the typical number density in interstellar space is included (e.g. $n \sim 3 \times 10^4\,\mathrm{m}^{-3}$), then an estimate of the distance to the pulsar can be made.

6.5 Faraday rotation

In treating the effect of EM waves on electrons we have ignored the magnetic force. This is no longer appropriate if there is a background magnetic field $\mathbf{B}_0$ in the plasma. Introducing a magnetic field brings a new characteristic frequency, the *cyclotron frequency*, which is Equation (6.20) in the limit of nonrelativistic motion:

$$\omega_B = \frac{eB_0}{m}. \tag{6.65}$$

The magnetic field also defines a preferred direction, and it turns out that the propagation of waves depends on their direction with respect to $\mathbf{B}_0$. In the following discussion we restrict attention to the case of waves propagating along the magnetic field.

The equation of motion of an electron in the plasma can now be written

$$m\frac{d\mathbf{v}}{dt} = -e\mathbf{E} - e\mathbf{v} \times \mathbf{B}_0. \tag{6.66}$$

We assume the magnetic field is oriented along the z axis,

$$\mathbf{B}_0 = B_0\widehat{\mathbf{z}}. \tag{6.67}$$

The effect we are interested in here is *Faraday rotation*, which occurs because left-hand and right-hand circularly polarised waves propagate with different speeds along a magnetic field. Hence we introduce the electric field at a point produced by a circularly polarised wave propagating in the z direction (see Problem Set 6):

$$\mathbf{E}(t) = Ee^{-i\omega t}(\widehat{\mathbf{x}} \mp i\widehat{\mathbf{y}}) = \widetilde{\mathbf{E}}e^{-i\omega t}. \tag{6.68}$$

The minus sign corresponds to a RH wave, and the plus sign to a LH wave. Correspondingly we assume the velocity variation has the same form,

$$\mathbf{v}(t) = ve^{-i\omega t}(\widehat{\mathbf{x}} \mp i\widehat{\mathbf{y}}) = \widetilde{\mathbf{v}}e^{-i\omega t}. \tag{6.69}$$

Substituting Equations (6.67), (6.68) and (6.69) into (6.66) leads to

$$\widetilde{\mathbf{v}} = \frac{-ie}{m(\omega \pm \omega_B)}\widetilde{\mathbf{E}}, \tag{6.70}$$

where the upper sign applies for RH waves and the lower for LH waves. The current in the plasma is $\widetilde{\mathbf{J}} = -ne\widetilde{\mathbf{v}} = \sigma\widetilde{\mathbf{E}}$, which allows identification of the conductivity,

$$\sigma = \frac{ine^2}{m(\omega \pm \omega_B)}. \tag{6.71}$$

From the definition of the dielectric constant (6.50) it follows that

$$\epsilon_\pm = 1 - \frac{\omega_p^2}{\omega(\omega \pm \omega_B)}. \tag{6.72}$$

From the definition of the refractive index (6.58) it follows that RH and LH circularly polarised waves travel with different phase speeds. This result might be thought to be a curiosity, but it is not, because a plane polarised wave can be decomposed into a linear superposition of a RH and a LH circularly polarised wave. These components of the wave have different phase speeds, and the result is that the wave does not keep a constant plane of polarisation, but has an electric field vector that rotates as the wave propagates. This effect is Faraday rotation.

To estimate the magnitude of this effect, note that the angle through which a circularly polarised wave rotates in propagating a distance d is

$$\phi_\pm = \int_0^d k_\pm ds, \tag{6.73}$$

where

$$k_\pm = \frac{\omega}{c}\sqrt{\epsilon_\pm} \approx \frac{\omega}{c}\left[1 - \frac{1}{2}\left(\frac{\omega_p}{\omega}\right)^2\left(1 \mp \frac{\omega_B}{\omega}\right)\right], \tag{6.74}$$

applying (6.72) in the limit $\omega \gg \omega_p, \omega_B$. For a plane polarised wave, the angle of rotation is $\Delta\theta = \frac{1}{2}(\phi_+ - \phi_-)$, and substituting the approximation (6.74) leads to

$$\Delta\theta = \frac{1}{2\omega^2 c}\int_0^d \omega_p^2 \omega_B \, ds = \frac{e^3}{2mc\epsilon_0\omega^2}\int_0^d nB_{\parallel} ds. \tag{6.75}$$

Equation (6.75) has been derived for a magnetic field along the line of sight, but in fact it holds in general with $B_{\parallel}$ interpreted as the component of the field along the line of sight.

Since $\Delta\theta$ varies with frequency, measurements at different frequencies of the plane of polarisation of polarised sources can in principle provide information about the magnetic field along the line of sight (subject to an assumption about the plasma number density). However, the interstellar magnetic field is believed to change direction often, and so this method provides only a lower bound to actual field strengths.

Problem Set 6

1. Of the pulsars known in 1972, the largest dispersion measure was 400 pc.cm^{-3}. If the average interstellar electron density is $\approx 3 \times 10^4$ m^{-3}, is this pulsar likely to be in our galaxy?

2. Demonstrate that:

   (a) Equation (6.68) represents RH and LH circularly polarised radiation,

   (b) a plane polarised EM wave can be written as the superposition of two circularly polarised waves.

PHYS377 Astrophysics 2001
Assignment 1 due Friday March 23

1. The flux of sunlight at the earth's surface is 1.4 $\mathrm{kWm^{-2}}$. What would be the radiation pressure due to this flux on
   (a) a reflecting surface?
   (b) an absorbing surface?

2. A *pinhole camera* consists of a small circular hole in a box, as shown in Figure 1.

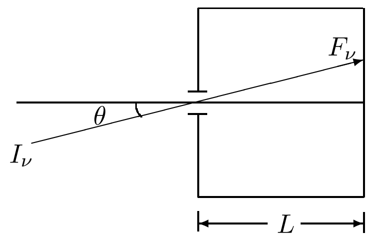


Figure 1: A pinhole camera.

The hole has diameter d and is a distance L from the film plane. Show that, provided d is small, the flux F_ν at the film plane depends on the brightness field $I_\nu(\theta, \phi)$ according to the approximate relationship

$$F_\nu \approx \frac{\pi \cos^4 \theta}{4 f^2} I_\nu(\theta, \phi),$$

where the *focal ratio* f is L/d. (Hence a pinhole camera provides a simple method for measuring I_ν.)

3. An optically thick sphere with radius R has temperature T_0 and emits thermally. It is surrounded by a shell of thermally emitting material at a temperature T_1 and with a thickness $x \ll R$, as shown in Figure 2. The shell of material has an absorption coefficient α_ν that is large near a frequency ν_0 and small otherwise, as shown in the inset diagram. The width of this absorption feature is small compared with $k_B T_i/h$ $(i = 0, 1)$. There are no other sources of emission or absorption, and scattering can be neglected.

   Consider two rays A and B received by a distant observer. Ray A comes from the centre of the sphere and ray B is just outside the limb of the sphere. The intensity $I_{\nu,A}$ is the intensity received by the observer along ray A, and $I_{\nu,B}$ is the intensity received along B.

   (a) Write down expressions for $I_{\nu,A}$ and $I_{\nu,B}$ in terms of the given variables.
   (b) Sketch $I_{\nu,A}$ and $I_{\nu,B}$ as functions of ν
      i. when $T_1 < T_0$,
      ii. when $T_1 > T_0$.

4. The spectrum of the Sun is characterised by dark lines called the Fraunhofer lines: emission in these lines is less than emission at neighbouring frequencies, and the lines are said to be *in absorption*. During solar eclipses bright colours are seen

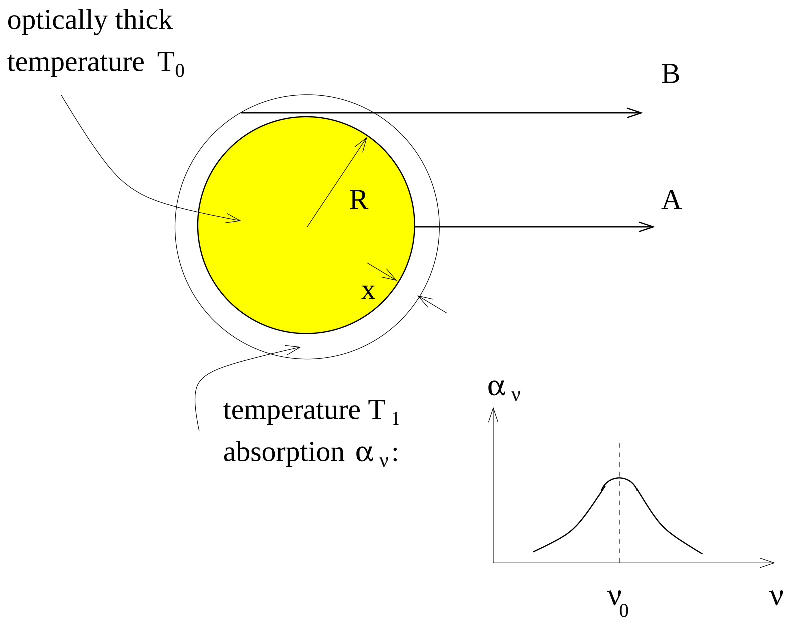


Figure 2: Details for Question 3.

from the limb of the Sun: spectra reveal that the frequency of emission matches the Fraunhofer lines, and the apppearance of colours indicates that emission in the lines is greater than at neighbouring frequencies, so that the Fraunhofer lines are being seen *in emission.* The region producing the colours during eclipses is the *chromosphere*, a thin layer of the solar atmosphere above the photosphere.

Explain:

(a) the appearance of the Fraunhofer lines in absorption,

(b) the appearance of the Fraunhofer lines in emission during solar eclipses.

PHYS377 Astrophysics 2001
Assignment 2 due Monday April 30

1. The equation of motion of a bound electron that is harmonically driven by an EM wave is
$$m\ddot{x} + m\omega_0^2 x = -eE_0 \cos\omega t,$$
where m is the electron mass, x describes the position of the electron, ω_0 is the natural frequency of oscillation of the electron due to its binding, E_0 is the amplitude of the driving electric field, and ω is the driving frequency.

   (a) Show that $x = x_0 \cos\omega t$ is the steady state solution, with
   $$x_0 = \frac{eE_0/m}{\omega^2 - \omega_0^2}.$$

   (b) Using the dipole approximation, show that the time averaged total power radiated by the electron is
   $$P = \frac{e^4 E_0^2}{12\pi\epsilon_0 m^2 c^3} \frac{\omega^4}{(\omega^2 - \omega_0^2)^2}.$$

   (c) The total cross section $\sigma(\omega)$ for scattering is defined by $P = \mathcal{F}\sigma(\omega)$, where $\mathcal{F} = \frac{1}{2}c\epsilon_0 E_0^2$ is the time averaged flux of the incident electromagnetic wave driving the system. Show that
   $$\sigma(\omega) = \frac{\sigma_T}{[1 - (\omega_0/\omega)^2]^2},$$
   where σ_T is the Thomson cross section.

   (d) Determine the limiting forms of $\sigma(\omega)$ for $\omega \ll \omega_0$ and $\omega \gg \omega_0$. As discussed in the lectures, what do these cases correspond to?

2. Assume that an observer at rest with respect to the fixed distant stars sees an isotropic distribution of stars, i.e. in any solid angle $d\Omega$ the observer sees $dN = N d\Omega/(4\pi)$ stars, where N is the total number of stars.

   Suppose that a second observer (whose rest frame is K') is moving with a relativistic velocity $\beta = v/c$ in the x direction.

   (a) What is the distribution of stars seen by the moving observer? In other words, what is the distribution function $P(\theta', \phi')$ such that the number of stars in the solid angle $d\Omega'$ is $P(\theta', \phi')d\Omega'$?

   (b) Show that $P(\theta', \phi') = N/(4\pi)$ when $\beta = 0$.

   (c) Show that $\int P(\theta', \phi')d\Omega' = N$.

   (d) Sketch the angular distribution of stars seen by the moving observer. In what direction do they "bunch up"? [In drawing this diagram, think carefully about which direction $\theta' = 0$ corresponds to.]

3. Consider a sphere of ionized hydrogen plasma undergoing gravitational collapse. The sphere can be assumed to be at a constant isothermal temperature T_0 and to have a constant density (total mass M_0) during the time the sphere is observed. The radius of the sphere $R(t)$ is a decreasing function of time. The sphere emits thermal bremsstrahlung radiation.

(a) What is the total luminosity (power output) of the sphere as a function of M_0, $R(t)$ and T_0 assuming the sphere is optically thin?

(b) At time t_1 during the observing period the sphere becomes optically thick. Obtain an expression for the total luminosity of the sphere after that time, in terms of $R(t)$ and T_0.

(c) Sketch the luminosity as a function of time. Based on this graph, arrive at an implicit relationship, in terms of $R(t_1)$, for the time t_1 when the sphere became optically thick.

4. The Crab Nebula emits synchrotron radiation. Taking the magnetic field in some of the bright structures ('filaments') observed in the nebula to be about 10^{-4} Gauss (1 Gauss $\equiv 10^{-4}$ Tesla), show that:

(a) a non-relativistic electron in the filaments radiates at about 300 Hz, independent of energy,

(b) 10^9eV and 10^{12}eV electrons in the filaments radiate in the radio and visible regions respectively.

PHYS377 Astrophysics 2001
Assignment 1 due Friday March 23

1. The flux of sunlight at the earth's surface is 1.4 $\mathrm{kWm^{-2}}$. What would be the radiation pressure due to this flux on
   (a) a reflecting surface?
   (b) an absorbing surface?

2. A *pinhole camera* consists of a small circular hole in a box, as shown in Figure 1.

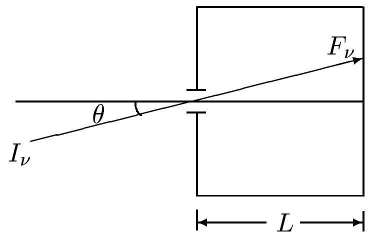


Figure 1: A pinhole camera.

The hole has diameter d and is a distance L from the film plane. Show that, provided d is small, the flux F_ν at the film plane depends on the brightness field $I_\nu(\theta, \phi)$ according to the approximate relationship

$$F_\nu \approx \frac{\pi \cos^4 \theta}{4f^2} I_\nu(\theta, \phi),$$

where the *focal ratio* f is L/d. (Hence a pinhole camera provides a simple method for measuring I_ν.)

3. An optically thick sphere with radius R has temperature T_0 and emits thermally. It is surrounded by a shell of thermally emitting material at a temperature T_1 and with a thickness $x \ll R$, as shown in Figure 2. The shell of material has an absorption coefficient α_ν that is large near a frequency ν_0 and small otherwise, as shown in the inset diagram. There are no other sources of emission or absorption, and scattering can be neglected.

   Consider two rays A and B received by a distant observer. Ray A comes from the centre of the sphere and ray B is just outside the limb of the sphere. The intensity $I_{\nu,A}$ is the intensity received by the observer along ray A, and $I_{\nu,B}$ is the intensity received along B.

   (a) Write down expressions for $I_{\nu,A}$ and $I_{\nu,B}$ in terms of the given variables.
   (b) Sketch $I_{\nu,A}$ and $I_{\nu,A}$ as functions of ν
      i. when $T_1 < T_0$,
      ii. when $T_1 > T_0$.

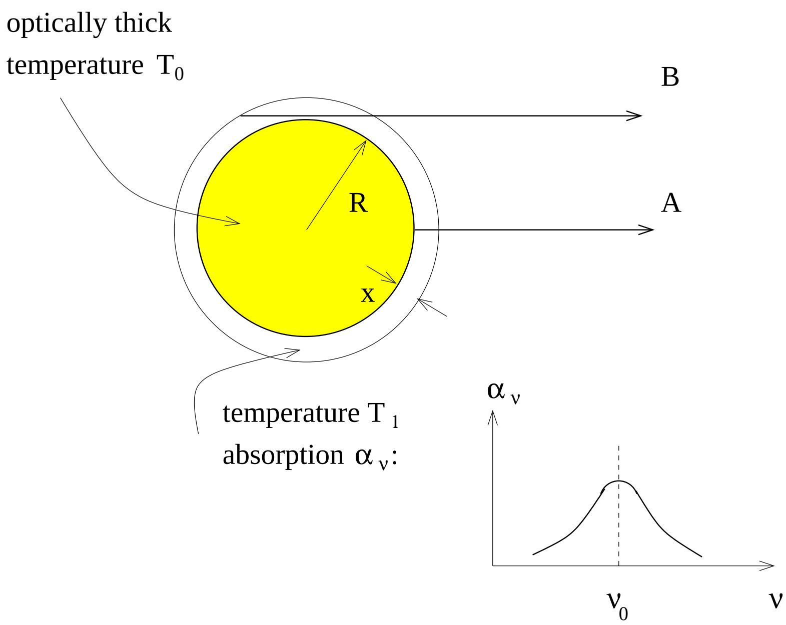


Figure 2: Details for Question 3.

4. The spectrum of the Sun is characterised by dark lines called the Fraunhofer lines: emission in these lines is less than emission at neighbouring frequencies, and the lines are said to be *in absorption*. During solar eclipses bright colours are seen from the limb of the Sun: spectra reveal that the frequency of emission matches the Fraunhofer lines, and the apppearance of colours indicates that emission in the lines is greater than at neighbouring frequencies, so that the Fraunhofer lines are being seen *in emission*. The region producing the colours during eclipses is the *chromosphere*, a thin layer of the solar atmosphere above the photosphere.

   Explain:

   (a) the appearance of the Fraunhofer lines in absorption,

   (b) the appearance of the Fraunhofer lines in emission during solar eclipses.

PHYS377 Astrophysics 2001
Assignment 2 due Monday April 30

1. The equation of motion of a bound electron that is harmonically driven by an EM wave is

$$m\ddot{x} + m\omega_0^2 x = -eE_0 \cos\omega t,$$

where m is the electron mass, x describes the position of the electron, ω_0 is the natural frequency of oscillation of the electron due to its binding, E_0 is the amplitude of the driving electric field, and ω is the driving frequency.

   (a) Show that $x = x_0 \cos\omega t$ is the steady state solution, with

$$x_0 = \frac{eE_0/m}{\omega^2 - \omega_0^2}.$$

   (b) Using the dipole approximation, show that the time averaged total power radiated by the electron is

$$P = \frac{e^4 E_0^2}{12\pi\epsilon_0 m^2 c^3} \frac{\omega^4}{(\omega^2 - \omega_0^2)^2}.$$

   (c) The total cross section $\sigma(\omega)$ for scattering is defined by $P = \mathcal{F}\sigma(\omega)$, where $\mathcal{F} = \frac{1}{2}c\epsilon_0 E_0^2$ is the time averaged flux of the incident electromagnetic wave driving the system. Show that

$$\sigma(\omega) = \frac{\sigma_T}{[1 - (\omega_0/\omega)^2]^2},$$

   where σ_T is the Thomson cross section.

   (d) Determine the limiting forms of $\sigma(\omega)$ for $\omega \ll \omega_0$ and $\omega \gg \omega_0$. As discussed in the lectures, what do these cases correspond to?

2. Assume that an observer at rest with respect to the fixed distant stars sees an isotropic distribution of stars, i.e. in any solid angle $d\Omega$ the observer sees $dN = N d\Omega/(4\pi)$ stars, where N is the total number of stars.

   Suppose that a second observer (whose rest frame is K') is moving with a relativistic velocity $\beta = v/c$ in the x direction.

   (a) What is the distribution of stars seen by the moving observer? In other words, what is the distribution function $P(\theta', \phi')$ such that the number of stars in the solid angle $d\Omega'$ is $P(\theta', \phi')d\Omega'$?

   (b) Show that $P(\theta', \phi') = N/(4\pi)$ when $\beta = 0$.

   (c) Show that $\int P(\theta', \phi')d\Omega' = N$.

   (d) Sketch the angular distribution of stars seen by the moving observer. In what direction do they "bunch up"? [In drawing this diagram, think carefully about which direction $\theta' = 0$ corresponds to.]

3. Consider a sphere of ionized hydrogen plasma undergoing gravitational collapse. The sphere can be assumed to be at a constant isothermal temperature T_0 and to have a constant density (total mass M_0) during the time the sphere is observed. The radius of the sphere $R(t)$ is a decreasing function of time. The sphere emits thermal bremsstrahlung radiation.

   (a) What is the total luminosity (power output) of the sphere as a function of M_0, $R(t)$ and T_0 assuming the sphere is optically thin?

   (b) At time t_1 during the observing period the sphere becomes optically thick. Obtain an expression for the total luminosity of the sphere after that time, in terms of $R(t)$ and T_0.

   (c) Sketch the luminosity as a function of time. Based on this graph, arrive at an implicit relationship, in terms of $R(t_1)$, for the time t_1 when the sphere became optically thick.

4. The Crab Nebula emits synchrotron radiation. Taking the magnetic field in some of the bright structures ('filaments') observed in the nebula to be about 10^{-4} Gauss ($1\,\text{Gauss} \equiv 10^{-4}$ Tesla), show that:

   (a) a non-relativistic electron in the filaments radiates at about 300 Hz, independent of energy,

   (b) 10^9eV and 10^{12}eV electrons in the filaments radiate in the radio and visible regions respectively.

MACQUARIE UNIVERSITY

Mid-year Examination 2001

Unit:	**PHYS 377 – ASTROPHYSICS I**
Date:	Friday 22 June, 9.20 am
Time Allowed:	**THREE** (3) hours, plus TEN (10) minutes reading time.
Total Number of Questions:	**TEN** (10).
Instructions:	Attempt **SIX** (6) questions only, **THREE** (3) from **EACH** of Parts A and B. Answer questions from Parts A and B in separate books. All questions are of equal value. Electronic calculators may be used, excepting those with a full alphabetic keyboard.

You may find the following information useful:

Boltzmann's constant:	$k = 1.381 \times 10^{-23}\,\mathrm{J\,K^{-1}}$
Planck's constant:	$h = 6.626 \times 10^{-34}\,\mathrm{J\,s}$
Speed of light:	$c = 2.998 \times 10^{8}\,\mathrm{m\,s^{-1}}$

$$P = \frac{1}{4\pi\epsilon_0}\frac{2(\ddot{p})^2}{3c^3}$$

$$\mathbf{p} = \sum_i q_i \mathbf{r}_i$$

$$P = \frac{1}{4\pi\epsilon_0}\frac{2q^2}{3c^3}\gamma^4(a_\perp^2 + \gamma^2 a_{||}^2)$$

$$\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma(t - vx/c^2) \end{aligned}$$

$$\gamma = (1 - v^2/c^2)^{-1/2}$$

$$dP = -\frac{\rho\, GM(r)\, dr}{r^2}$$

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + ...$$

$$\mu\frac{dI_\nu}{d\tau_\nu} = I_\nu - S_\nu$$

$$P_\nu = \frac{1}{c}\int I_\nu \cos^2\theta\, d\Omega$$

$$\pi F_\nu = \int I_\nu \cos\theta\, d\Omega$$

$$u_\nu = \frac{1}{c}\int I_\nu\, d\Omega$$

$$\pi F = \sigma T_{\mathrm{eff}}^4$$

$$B(T) = \frac{\sigma}{\pi}T^4$$

$$P = \frac{4\sigma}{3c}T^4$$

$$I_\nu(\tau_\nu) = I_0 e^{-\tau_\nu} + \int_0^{\tau_\nu}\frac{j_\nu}{k_\nu}e^{(\tau_\nu' - \tau_\nu)}d\tau_\nu'$$

$$B_\nu(T) = \frac{2h\nu^3}{c^2}\left[e^{h\nu/kT} - 1\right]^{-1}$$

$$\frac{N_n}{N_m} = \frac{g_n}{g_m}e^{-(E_n - E_m)/kT}$$

$$Z(T) = \sum_n g_n e^{-E_n/kT}$$

$$\frac{N_i}{N_n} = 2\frac{g_i}{g_n}\frac{(2\pi m_e kT)^{3/2}}{n_e h^3}e^{-\chi_n/kT}$$

$$\frac{N_i}{N_0} = 2\frac{Z_i}{Z_0}\frac{(2\pi m_e kT)^{3/2}}{n_e h^3}e^{-\chi_0/kT}$$

$$x = \frac{N_i}{N_0 + N_i}$$

Part A

Attempt THREE (3) questions from Part A
(60 marks in total, all questions are of equal value)
Answer questions from Part A in a separate book

1. (a) (8 marks)
The equation

$$\mathcal{F}_\nu = \int I_\nu \cos\theta \, d\Omega \tag{1}$$

relates intensity I_ν ($\mathrm{Wm^{-2}Hz^{-1}ster^{-1}}$) to flux $\mathcal{F}_\nu$ ($\mathrm{Wm^{-2}}$).
Use (1) to show that the flux $\mathcal{F}_0$ at a distance r from an infinite plane with a uniform brightness B_0 is

$$\mathcal{F}_0 = \pi B_0,$$

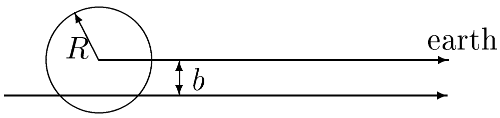
independent of the distance r.

(b) (2 marks)
One version of the radiative transfer equation is

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu. \tag{2}$$

For $\alpha_\nu = 0$, write down the solution to this equation.

(c) (10 marks)
A spherical cloud of gas has radius R, temperature T and is a distance d from Earth ($d \gg R$). The cloud emits thermally at a rate $P_\nu = 4\pi j_\nu$ (this is the power output per unit volume and per unit frequency range) and the cloud is optically thin.



i. What is the brightness of the cloud, as measured from Earth. Give the answer as a function of distance b from the cloud centre, assuming that the cloud is viewed along rays parallel to a line to its centre (see diagram).
ii. What is the flux of the whole cloud, as measured at Earth?
iii. If the cloud were optically thick, what would the answer to (c) ii. be?

2. (a) (5 marks)

A positive charge is moving with a (constant) relativistic speed along the x axis and then decelerates to rest, ending up at rest at the position x_0 at time t_0. The deceleration takes a short time Δt.

Sketch a 2-D diagram of the electric field lines of the charge at a time $t_1 > t_0$, including showing the field at a distance $r > c(t_1 - t_0 + \Delta t)$ from the charge. Indicate the position x_1 that the charge would have reached at time t_1 if it had not decelerated.

(b) (5 marks)

The displacement of a point charge q oscillating with a fixed amplitude x_0 and a given frequency ω may be written

$$x = x_0 \cos(\omega t).$$

Use the dipole approximation to obtain an expression for the time-averaged power emitted by the charge.

(c) (5 marks)

Consider a free electron oscillating in response to the passage of an electromagnetic wave with electric field

$$\mathbf{E} = E_0 \cos(\omega t)\hat{\mathbf{z}}$$

in the vicinity of the electron.

Use the dipole approximation to obtain an expression for the time-averaged power emitted by the electron.

(d) (5 marks)

In about half a page, describe an example of the scattering process described by your result in (b) OR describe an astrophysical example of the scattering process described by your result in (c).

3. (a) (7 marks)

Show that the relationship between velocities $\mathbf{u}$ and $\mathbf{u}'$ measured in frames K and K' (frame K' moves with respect to K at speed v in the $+x$ direction) may be written

$$u_{\parallel} = \frac{u'_{\parallel} + v}{1 + vu'_{\parallel}/c^2}, \qquad u_{\perp} = \frac{u'_{\perp}}{\gamma(1 + vu'_{\parallel}/c^2)},$$

where the subscripts $\parallel$ and $\perp$ label components parallel and perpendicular to the relative motion of the frames, and where $\gamma = (1 - v^2/c^2)^{-1/2}$.

(b) (5 marks)

If θ and θ' denote the measured angles of propagation of a photon with respect to the x and x' axes in the two frames, show that

$$\cos\theta = \frac{\cos\theta' + v/c}{1 + (v/c)\cos\theta'}, \qquad \sin\theta = \frac{\sin\theta'}{\gamma\,[1 + (v/c)\cos\theta']}. \tag{3}$$

(c) (5 marks)

Setting $\theta' = \pi/2$ in (3) gives

$$\sin\theta = \frac{1}{\gamma}. \tag{4}$$

Use (4) to explain, in about half a page, the phenomenon of *beaming*, i.e. that a highly relativistic particle emits radiation in a narrow cone in its direction of motion.

(d) (3 marks)

Sketch the radiation pattern produced by a non-relativistic particle with acceleration $\mathbf{a}$ (indicate $\mathbf{a}$ in your diagram), and sketch the radiation pattern produced by a highly relativistic particle with velocity $\mathbf{v}$ and acceleration $\mathbf{a}$, where $\mathbf{v} \parallel \mathbf{a}$.

4. The relativistic equation of motion of an electron in a magnetic field $\mathbf{B}$ may be written

$$\frac{d}{dt}(\gamma m_0 \mathbf{v}) = -e\mathbf{v} \times \mathbf{B},$$

in the notation of the lectures.

(a) (3 marks)
Why is γ a constant for the electron?

(b) (3 marks)
Write down equations describing the rate of change of the components of velocity $\mathbf{v}_{||}$ and $\mathbf{v}_{\perp}$ parallel and perpendicular to the magnetic field.

(c) (5 marks)
Summarise the argument that the electron undergoes helical motion around a magnetic field line. What is the frequency ω_B of the motion perpendicular to the magnetic field, in terms of e, B, γ and m_0?

(d) (4 marks)
Write down the acceleration $a_{\perp}$ of the electron. Use the relativistic Larmor formula to derive an expression for the total power radiated by the electron. What is this radiation called?

(e) (5 marks)
For a non-relativistic particle, the radiation is produced at the frequency ω_B. For a relativistic particle, the radiation is no longer at a single frequency. In about half a page give a qualitative explanation of why a relativistic particle produces a broad spectrum of radiation (you do not need to derive an expression for the range of frequencies emitted by the electron).

5. An astronomical object moves with relativistic velocity v at an angle θ to the line of sight to a distant observer.

(a) (6 marks)
Show that the apparent transverse velocity of the object is

$$v_{\rm app} = \frac{v \sin\theta}{1 - (v/c)\cos\theta}.$$

(b) (6 marks)
For a fixed v, find the angle at which $v_{\rm app}$ is a maximum. Hence show that $v_{\rm app}$ has the maximum value γv, where $\gamma = (1 - v^2/c^2)^{-1/2}$ is the Lorentz factor of the object. It follows that $v_{\rm app}$ can exceed c. Why is this not a problem?

(c) (8 marks)
Suppose the object is emitting radiation with a frequency ω', measured in the rest frame of the object. Show that the frequency of radiation received by the distant observer is

$$\omega = \frac{\omega'}{\gamma\,[1 - (v/c)\cos\theta]}.$$

Part B

Attempt THREE (3) questions from Part B
(60 marks in total, all questions are of equal value)
Answer questions from Part B in a separate book

6. Consider a star with a linear variation in gas density,

$$\rho = \rho_c \left(1 - \frac{r}{R}\right),$$

where ρ_c is the central density and R is the radius of the star.

Show that for this star:

(a) (6 marks)
the mass varies with radius according to

$$m(r) = M\left(\frac{4r^3}{R^3} - \frac{3r^4}{R^4}\right),$$

where M is the total mass of the star,

(b) (6 marks)
the pressure varies with radius according to

$$P(r) = \frac{5\pi}{36} G\rho_c^2 R^2 \left(1 - \frac{24}{5}\frac{r^2}{R^2} + \frac{28}{5}\frac{r^3}{R^3} - \frac{9}{5}\frac{r^4}{R^4}\right),$$

(c) (8 marks)
under ideal gas conditions, and in the absence of radiation, the temperature varies according to

$$T(r) = \frac{5\pi}{36}\frac{G\mu m_H}{k}\rho_c R^2 \left(1 + \frac{r}{R} - \frac{19}{5}\frac{r^2}{R^2} + \frac{9}{5}\frac{r^3}{R^3}\right).$$

7. (a) (2 marks)
Define the optical depth of a gas. Show how it is related to the volume opacity of the gas.

(b) (6 marks)
Show that if the source function is isotropic the equation of radiative transfer can be written

$$\pi F = -\frac{c}{\overline{k}}\frac{dP_R}{ds},$$

where $\overline{k}$ is the Rosseland mean opacity.

(c) (8 marks)
Show that if there is thermodynamic equilibrium then

$$\pi F = \frac{4ac}{3\overline{k}}T^3\frac{dT}{dr},$$

where $a = 4r/c$.

(d) (4 marks)
Show that under the above conditions, high energy photons ($h\nu \sim 4kT$) are the prime source of heat conduction.

8. (a) (6 marks)
   List some of the different types of temperature that can be used to describe a system that is not in thermodynamic equilibrium. For each, outline the phenomenon to which it is related.

   (b) (8 marks)
   Starting from Boltzmann's law, outline the derivation of the Saha equation. Explain its significance.

   (c) (4 marks)
   Using the Saha equation explain:

      i. the importance of some trace elements in determining the degree of ionisation of hydrogen at a given temperature,

      ii. the variations in temperature between giant and dwarf stars of the same spectral class.

   (d) (2 marks)
   Sketch how the degree of ionisation of a given species varies with temperature. Relate this to the Saha equation.

9. (a) (4 marks)
Outline the various processes that affect the width of spectral lines.

(b) (2 marks)
Define the rectified line profile and show its functional dependence on the line opacity.

(c) (2 marks)
Define the equivalent width of a spectral line.

(d) (6 marks)
Describe the growth of a line and indicate the processes responsible for its shape.

(e) (6 marks)
Indicate how the excitation temperature of a gas can be determined from observations of the curves of growth for two series of lines.

10. (a) (6 marks)

Show that in the atmosphere of a star the temperature is given by

$$T^4 = \frac{3}{4} T_{\text{eff}}^4 \left(\tau + \frac{2}{3} \right).$$

(b) (8 marks)

Show that if the source function is linear with optical depth,

$$S_\nu = a + b\tau_\nu$$

then the intensity $I_\nu(\mu)$ as a function of $\mu = \cos\theta$ is given by

$$I_\nu(\theta) = a + b\mu.$$

Hence show that the limb darkening law is

$$\frac{I(r)}{I(0)} = 1 - \beta + \frac{\beta}{R}\sqrt{R^2 - r^2}$$

where $\beta = b/(a+b)$, and where R is the radius of the star and r is the radial distance from the centre of the visible disk.

(c) (6 marks)

Describe briefly how to investigate the temperature structure of an atmosphere if the limb darkening law is known.