Quantum Non-Locality and Latent Causal Structures

David Gross
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Coogee (yeah!)
Jan 2017
Outline

Outline:

▶ Background
▶ Entropies
▶ LPs
▶ SDPs

People:

▶ Rafael Chaves,
  R. Kueng, C. Majenz,
  L. Luft, A. Kela,
  K. Prillwitz, J. Aberg,
  D. Janzing,
  B. Schollköpf.

R. Chaves et al., Uncertainty in Artificial Intelligence 2014
Outline

Not recent enough?

David Gross
(University of Freiberg, Germany)

Non-Negative Phase Space Distributions, with the Benefit of Hindsight

R. Chaves et al., Uncertainty in Artificial Intelligence 2014
Testing causal structures

I used to think correlation implied causation.

Then I took a statistics class. Now I don’t.

Sounds like the class helped. Well, maybe.
Is obesity contagious?

Empirical finding: People of similar weight more likely to be friends.
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Various possible explanations:

- Prefer friends with similar body constitution.
Is obesity contagious?

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Various possible explanations:

▶ Prefer friends with similar body constitution.
▶ Immitate eating habits of friends.
▶ “Obesity is contagious”
Is obesity contagious?

Empirical finding: People of similar weight more likely to be friends.

Various possible explanations:

- Prefer friends with similar body constitution.
- Immitate eating habits of friends.
- “Obesity is contagious”
- Unobserved common cause.
Causal relationships can be probed by *interventions*:

Compare

\[
\Pr[\text{friends} \mid \text{same weight}]
\]

\[
\Pr[\text{friends} \mid \text{do}(\text{same weight})]
\]

\[
\Pr[\text{do(friends)} \mid \text{weight}].
\]
Passive Causal Inference?

However:

- Interventions often impractical / unethical

Natural Question:

Can one obtain information about causal relations from empirical observations?
Causal structures

To address problem, formalize notions:

- For $n$ variables $X_1, \ldots, X_n$,
- a causal structure or Bayesian network is directed acyclic graph,
- with $i$th variable function

$$X_i = f_i(pa_i, u_i)$$

of its parents $pa_i$ and “local randomness” $u_i$. 
Causal structures

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- For $n$ variables $X_1, \ldots, X_n$,
- a causal structure or Bayesian network is directed acyclic graph,
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\[ X_i = f_i(p_{a_i}, u_i) \]

of its parents $p_{a_i}$ and “local randomness” $u_i$.

Chain rule of probability $\Rightarrow p(x_1, \ldots, x_i) = \prod_{i=1}^{n} P(x_i|p_{a_i}, u_i)$. 
Causal structures

To address problem, formalize notions:

- For \( n \) variables \( X_1, \ldots, X_n \),
- a causal structure or Bayesian network is directed acyclic graph,
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  \[
  X_i = f_i(p_{a_i}, u_i)
  \]
  of its parents \( p_{a_i} \) and “local randomness” \( u_i \).

Chain rule of probability \( \Rightarrow \) \( p(x_1, \ldots, x_i) = \prod_{i=1}^{n} P(x_i|p_{a_i}, u_i) \).

In TN language: Contraction of (non-negative) tree tensor network.
Local Markov Condition

Causal structure does imply testable conditions

Ex.: “wiping” independent of “cold” conditioned on “sneezing”.

Result:
1. All corollaries of causal structures follow from Local Markov Conditions.
2. Recoverable aspects of causality graph well-understood. [Pearl, 2000]
Local Markov Condition

Causal structure does imply testable conditions

Ex.:
- “wiping” independent of “cold” *conditioned* on “sneezing”.

More generally:
- $X_i$ is independent of its non-descendants, given its parents.
- “Local Markov Condition”.

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[Pearl, 2000]
Hidden variables (confounders / latent variables)

... however, analysis breaks down if only subset of variables accessible.

Ex.: “common ancestor” problem:

- Pair-wise structure implies no independences between $A$, $B$, $C$,
- but is not compatible, e.g. with 3 perfectly correlated coins.
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Ex.: “common ancestor” problem:

- Pair-wise structure implies no independences between $A$, $B$, $C$,
- but is not compatible, e.g. with 3 perfectly correlated coins.
- Amazingly, this example not yet fully characterized! (→ Later)
Independences $=$ algebraic constraints

$$p(x, y) = p(x)p(y)$$

$\iff$ \(\text{rank}(p(x, y)) = 1\)

Rank variety $+$ Positivity $=$ real algebraic geometry
Independences = algebraic constraints

\[ p(x, y) = p(x)p(y) \]

\[ \Leftrightarrow \quad \text{rank}(p(x, y)) = 1 \]

Rank variety + Positivity = real algebraic geometry

Nasty in theory and practice...

...so new ideas needed.
Diverse Applications...
Diverse Applications. . .
Diverse Applications...

Bell inequalities for social networks 09jun11

I’m happy to unveil a new paper, “A sequence of relaxations constraining hidden variable models”.

Depending on your interests, I’m including two different overviews. One comes from the social networks perspective and the other from the quantum physics perspective. Fundamentally, it’s about detecting hidden variables.
Entropic Marginals
1. Entropy cone

Step 1/3: The unconstrained, global object.

- Associate with $S \subset \{1, \ldots, n\}$ the joint entropy $S(X_S)$
- $\Rightarrow$ an *entropy vector* $v \in \mathbb{R}^{2^n}$, indexed by subsets

Ex.: $(H(\emptyset), H(A), H(B), H(A, B))$
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- Structure not fully understood, but...
- ... contained in Shannon cone cone $\Gamma_n$, defined by strong subadditivity and monotonicity.

\[
H(A, B) \leq H(A, B, C), \quad H(A, B) \leq H(A) + H(B), \quad I(B : C|A) \geq 0.
\]
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\[ H(A, B) \leq H(A, B, C), \quad H(A, B) \leq H(A) + H(B), \quad I(B : C | A) \geq 0. \]

- We will mostly work with Shannon relaxation.
2. Causal constraints

Step 2/3: Now choose candidate structure and add causal constraints.

That's easy: Conditional independences measured by mutual information: $I(\{\text{wipe}\};\{\text{hay, cold}\}|\text{sneeze}) = 0$.

Can even relax: $I(\{\text{wipe}\};\{\text{hay, cold}\}|\text{sneeze}) \leq \epsilon$.

$\Rightarrow$ cone $C$ of constraints.

$\Rightarrow$ new global cone $\Gamma_n \cap C$ of entropies subject to causal structure.
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- \[ \Rightarrow \text{cone} \ C \text{ of constraints.} \]

\[ \Rightarrow \text{new global cone} \ \Gamma_n \cap C \text{ of entropies subject to causal structure.} \]
3. Marginalize

Step 3/3: Marginalize.

- Set $\mathcal{M} \subset 2^{\{1, \ldots, n\}}$ of jointly observable r.v.’s is *marginal scenario*.

- Classically: r.v.’s either observable or not

QM: Some r.v.’s not jointly measurable.
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Marginalize to $\mathcal{M}$:

- Geometrically trivial: just restrict $\Gamma_n \cap C$ to observable coordinates.
- Algorithmically costly: $\Gamma_n \cap C$ represented in terms of inequalities (use, e.g. Fourier-Motzkin elimination)
3. Marginalize

Step 3/3: Marginalize.

- Set $\mathcal{M} \subset 2^{\{1,\ldots,n\}}$ of jointly observable r.v.’s is marginal scenario.

- Classically: r.v.’s either observable or not
  QM: Some r.v.’s not jointly measureable.

Marginalize to $\mathcal{M}$:

- Geometrically trivial: just restrict $\Gamma_n \cap C$ to observable coordinates.

- Algorithmically costly: $\Gamma_n \cap C$ represented in terms of inequalities (use, e.g. Fourier-Motzkin elimination)

Final result: description of marginal, causal, entropy cone $(\Gamma_n \cap C)_{|\mathcal{M}}$ in terms of “entropic Bell inequalities”.
1. Relation Entropy & Binary Bell Ineqs

Revisit “entropic CHSH” [Braunstein & Caves ’88 (!)]

\[
\begin{align*}
&\langle X_A X_B \rangle + \langle Y_A X_B \rangle + \langle Y_A Y_B \rangle - \langle X_A Y_B \rangle \leq \langle X_A \rangle + \langle X_B \rangle \\
&- H(X_A X_B) - H(Y_A X_B) - H(Y_A Y_B) + H(X_A Y_B) \geq -H(X_A) - H(X_B)
\end{align*}
\]

▶ Measures frustration in degree of correlation, rather than sign.
▶ Resembles “sign-reversed” CHSH. No coincidence...
1. Relation Entropy & Binary Bell Ineqs

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$$\langle X_A X_B \rangle + \langle Y_A X_B \rangle + \langle Y_A Y_B \rangle - \langle X_A Y_B \rangle \leq \langle X_A \rangle + \langle X_B \rangle$$

$$- H(X_A X_B) - H(Y_A X_B) - H(Y_A Y_B) + H(X_A Y_B) \geq -H(X_A) - H(X_B)$$

- Measures frustration in degree of correlation, rather than sign.
- Resembles “sign-reversed” CHSH. No coincidence. . .
- Result: Negative of any multipartite entropic ineq also valid for probabilities. [NJP ’13]
- Often, converse true ⇒ Source of entropic Bell ineqs [NJP ’13]
2. Common Ancestors & Strength of Causal Influence

- Entropic constraints given by (perms of)
  \[ B = I(A : B) + I(A : C) - H(A) \leq 0. \]

- Ex.: Perfectly correlated coins: \( B = 1. \)
2. Common Ancestors & Strength of Causal Influence

- Entropic constraints given by (perms of)

\[ \mathcal{B} = I(A : B) + I(A : C) - H(A) \leq 0. \]

- Example: Perfectly correlated coins: \( \mathcal{B} = 1. \)

- Violation \( \mathcal{B} \) interpretable as causal strength of direct influence \( A \to B \) required to explain data [UAI '14]

Def. causal strength \( C_{A \to B} \) as relative entropy distance incurred by cutting link.

Then \( C_{A \to B} \geq \mathcal{B} \). [UAI '14]
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- Entropic constraints given by \((\text{perms of})\)
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- Def. causal strength \(C_{A \rightarrow B}\) as relative entropy distance incurred by cutting link.

- Then \(C_{A \rightarrow B} \geq \mathcal{B}\). [UAI '14]
3. Many more...

Can treat...

- Scenarios of $n$ observables with independent common ancestors influencing at most $M$ each

- Direction of causation from pairwise marginals

...and more.

[UAI ’14]
Entropy & Quantum Causal Stuctures
With minor modifications, causal diagrams make sense for quantum systems.
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Nodes are states. Labels designate systems.

If node has incoming edges, state results from CP map applied to incoming systems.

Sample diagram says

\[
\rho_{ABC} = \left[ \Phi_{A_1 A_2 \rightarrow A} \otimes \Phi_{B_1 B_2 \rightarrow B} \otimes \Phi_{C_1 C_2 \rightarrow C} \right] (\rho_{A_1 B_2} \otimes \rho_{A_2 C_2} \otimes \rho_{B_2 C_2}).
\]
Quantum Causal Structures 2

How to build entropic constraints for quantum causal structures:

1. Use von Neumann entropy
   ⇒ drop monotonicity ineq. $H(A, B) \geq H(A)$
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1. Use von Neumann entropy
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2. QM does not assign joint state to input & output of operation! No \( H(A_1, A) \)!
   Consider only coexisting variables!
Quantum Causal Structures 2

How to build entropic constraints for quantum causal structures:
1. Use von Neumann entropy
   \[ H(A, B) \geq H(A) \]
   \[ \Rightarrow \text{drop monotonicity ineq.} \]
2. QM does not assign joint state to input & output of operation! No "\( H(A_1, A) \)"!
   Consider only \textit{coexisting variables}!
3. Use data processing inequality to relate non-coexisting variables. Ex.:
   \[ I(A : B) \leq I(A_1A_2 : B_1B_2). \]
How to build entropic constraints for quantum causal structures:

1. Use von Neumann entropy
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3. Use data processing inequality to relate non-coexisting variables. Ex.:
   \[ I(A : B) \leq I(A_1A_2 : B_1B_2). \]

...gives rich theory [Nat. Comm. ’14].
Quantum Causal Structures Ex.: Information Causality

Recall inf. caus. game: [Pawlowski et al., Nature ’09]

- Alices receives bits $X_1, \ldots, X_n$, sends message $M$ to Bob
- Bob recives $M$ and challenge $S \rightarrow$ outputs guess $Y$ for $X_S$
- Aided by joint quantum state $\rho_{AB}$
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Original inequality:

$$\sum_s I(X_s : Y|S = s) \leq H(M)$$

Strengthening using systematic “quantum causal structures” prot.:

$$I(X_1 : Y_1, M) + I(X_2 : Y_2, M) + I(X_1 : X_2|Y_2, M) \leq H(M) + I(X_1 : X_2).$$
Quantum Causal Structures Ex.: Information Causality

Two consequences of strengthened ineq.:

1. Violation measures “direct causal influence” $C_{X \rightarrow Y}$
Quantum Causal Structures Ex.: Information Causality

Two consequences of strengthened ineq.:

1. Violation measures “direct causal influence” $C_{X \rightarrow Y}$
2. Detects more post-quantum correlations:
LPs & relaxations of causal assumptions in Bell scenarios
Relaxations of causal assumptions in Bell scenarios

In this part:

▶ Do not work with entropies.

But show how...

▶ ...graphical notation of causality make it easy to reason about relaxations of causal assumptions.
▶ ...the idea of quantifying “causal influence” is fruitful for Bell scenarios.
Relaxations of causal assumptions in Bell scenarios

Constraints encoded by Bell causal structure have names:

- Locality

  \[ p(b|x, y, \lambda) = p(b|y, \lambda). \]

- Measurement independence

  \[ p(x, y, \lambda) = p(x)p(y)p(\lambda). \]
Relaxations of causal assumptions in Bell scenarios

Constraints encoded by Bell causal structure have names:

- **Locality**
  \[ p(b|x, y, \lambda) = p(b|y, \lambda). \]

- **Measurement independence**
  \[ p(x, y, \lambda) = p(x)p(y)p(\lambda). \]

How much do we need to relax the causal assumptions entering in Bell’s theorem to explain “non-local correlations” classically?
Relaxations

- **Ingredient 1:** More general *causal structures*

  ![Graphs](image)

  - (a) Bipartite Bell
    - \(X\) \(\rightarrow\) \(A\) \(\rightarrow\) \(\Lambda\) \(\rightarrow\) \(B\) \(\rightarrow\) \(Y\)
  - (b) Rel. of locality
    - \(X\) \(\rightarrow\) \(A\) \(\rightarrow\) \(\Lambda\) \(\rightarrow\) \(B\)
    - \(Y\) \(\rightarrow\) \(B\)
  - (c) Rel. of locality
    - \(X\) \(\rightarrow\) \(A\) \(\rightarrow\) \(\Lambda\) \(\rightarrow\) \(B\)
    - \(Y\) \(\rightarrow\) \(B\)
  - (d) General comm.
    - \(X\) \(\rightarrow\) \(A\) \(\rightarrow\) \(\Lambda\) \(\rightarrow\) \(B\) \(\rightarrow\) \(Y\)
    - \(M\) \(\rightarrow\) \(X\)
  - (e) Rel. of meas. ind.
    - \(X\) \(\rightarrow\) \(A\) \(\rightarrow\) \(\Lambda\) \(\rightarrow\) \(B\)
    - \(\mu\) \(\rightarrow\) \(\mu\)
    - \(Y\) \(\rightarrow\) \(Y\)

- **Ingredient 2:** Quantitative measures of *causal strength*
Relaxations

- Ingredient 1: More general *causal structures*
- Ingredient 2: Quantitative measures of *causal strength*

Meas. $C_{A\!\to\!B}$ used here: Maximal change in total variational distance incurred by manually changing $A$:

$$C_{A\!\to\!B} = \sup_{a,a'} \sum_\lambda p(\lambda)|p(b|\text{do}(a), \lambda) - p(b|\text{do}(a'), \lambda)|$$
Results

Main Result

Quantitative minimum of relaxation necessary to classically explain observed data can often be cast as a linear program. Moreover, closed-form results can often be obtained using duality theory.
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Quantitative minimum of relaxation necessary to classically explain observed data can often be cast as a linear program. Moreover, closed-form results can often be obtained using duality theory.

▶ Causal interpretation of numerical CHSH violation:

\[
\min C_{A \rightarrow B} = \min C_{X \rightarrow B} = \max \{0, \text{CHSH}\}
\]
Results

Main Result

Quantitative minimum of relaxation necessary to classically explain observed data can often be cast as a linear program. Moreover, closed-form results can often be obtained using duality theory.

- Quantitative bound on measurement dependence

\[
\min \mathcal{M} = \max\{0, I_d/4\},
\]

where

\[
\mathcal{M} = \| p(\lambda, x, y) - p(\lambda)p(x, y) \|_{TV}
\]

and \( I_d \) violation of CGLMP-inequality.
Results

Main Result

Quantitative minimum of relaxation necessary to classically explain observed data can often be cast as a linear program. Moreover, closed-form results can often be obtained using duality theory.

- Quantum violations even for classical models that allow for communication of measurement outcomes!
Recent Australian experiments

Experimental Test of Nonlocal Causality

M. Ringbauer$^{1,2}$, C. Giarmatzi$^{1,2}$, R. Chaves$^{3,4}$, F. Costa$^1$, A. G. White$^{1,2}$ & A. Fedrizzi$^{1,2,5}$

$^1$Centre for Engineered Quantum Systems, $^2$Centre for Quantum Computer and Communication Technology, School of Mathematics and Physics, University of Queensland, Brisbane, QLD 4072, Australia,
$^3$Institute for Physics & FDM, University of Freiburg, 79104 Freiburg, Germany, $^4$Institute for Theoretical Physics, University of Cologne, 50937 Cologne, Germany,
$^5$School of Engineering and Physical Sciences, SUPA, Heriot-Watt University, Edinburgh EH14 4AS, UK

[Science Advances ’16]
Semi-definite programming bounds
Recall we can’t even figure out triangle...
This might be a chance!

... side remark...
This might be a chance!

**Three qubits can be entangled in two inequivalent ways**
Abstract: Invertible local transformations of a multipartite system are used to define equivalence classes in the set of entangled states. This classification concerns the entanglement properties of a single copy of the state. Accordingly, we say that two states ...
Cited by 1683 - Related articles - BL Direct - All 22 versions - Import into BibTeX

**Four qubits can be entangled in nine different ways**
F Verstraete, J Dehaene, B De Moor... - Physical Review A, 2002 - APS
... to the singlet state by SLOCC operations 3. In the case of three entangled qubits, it was shown 2,4,5 that each state can be converted by SLOCC operations either to the GHZ-state (000 111)
\& or to the W-state (001 010 100 \rangle), leading to two inequivalent ways of entangling ...
Cited by 350 - Related articles - BL Direct - All 12 versions - Import into BibTeX

**Control and measurement of three qubit entangled states**
CF Roos, M Riebe, H Haffner, W Hansel... - Science, 2004 - sciencemag.org
... The ions' electronic qubit states are initialized with the S state by optical pumping. Three qubits can be entangled in only two inequivalent ways, represented by the Greenberger-Horne-Zeilinger (GHZ) state, \& the W state, (17). ...
Cited by 273 - Related articles - All 13 versions - Import into BibTeX
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Cited by 273 - Related articles - All 13 versions - Import into BibTeX

Three coins can be correlated in how many ways?
Recall we can’t even figure out triangle...
Recall we can’t even figure out triangle... 

... new outer approximations based covariances
Recall we can’t even figure out triangle…

…new outer approximations based \textit{covariances}

- Assume all observable quantities take values in a vector space.
- Q: What can we say about its covariance matrix?
SDPs

- SDP test more powerful than entropic ineqs. for triangle.
- ...but true transition point still not known.

Summary

- Causal structures and Bell nonlocality go well together
- Field relatively young – pick that fruit!